Aspects of Growth in Finitely Generated Groups

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Abstract

In this thesis we study various variants of word growth in finitely generated groups, focussing on conjugacy growth. For virtually abelian groups, we prove that the conjugacy growth series, coset growth series (for any subgroup) and relative growth series of any subgroup are rational for any choice of finite weighted generating set. We draw together work of Stoll and Babenko to produce asymptotic estimates of the conjugacy growth of class 2 nilpotent groups whose derived subgroup is infinite cyclic. These results have implications for the associated series. We also study the Baumslag-Solitar groups of the form $BS(1,k)$, proving that they have transcendental conjugacy growth series with respect to their standard generating sets, providing explicit formulae for their conjugacy growth series, and calculating their growth rates.

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Chapter 1

Introduction

This thesis is in the general area of geometric and combinatorial group theory, which grew out of the work of Dehn and others, who studied groups via their presentations by generators and relations. More recent approaches study groups by viewing them as metric spaces, or through their actions on other spaces. This is a rapidly growing area drawing together algebra, geometry, low-dimensional topology, dynamics, and formal language theory.

For a group presentation $G = \langle S \mid R \rangle$, we denote by S^* the set of words over the generators (i.e. the free monoid on S). We view the pair (G, S) as a metric space by constructing its *Cayley graph*, whose vertices are in bijection with the elements of the group, with edges between vertices whenever one element can be reached from the other via right multiplication by a generator. If the generating set S is finite and inverse-closed, we can consider the (undirected) graph with the natural metric where each edge has length 1.

A natural quantity to count is the number of group elements that can be 'spelled' with words in S^* of length up to n. Equivalently, this is the number of vertices in the metric ball of radius n in the Cayley graph. The *growth function* $\beta \colon \mathbb{N} \to \mathbb{N}$ encodes precisely this information. A loose interpretation is that groups with slower growth function are 'smaller than' groups with faster growth function (although all finitely generated infinite groups are countable). A key result is that, up to bounded scaling, the growth function is a group invariant: it does not depend on the choice of finite generating set. Furthermore, it is a geometric invariant in the sense that it is invariant under quasi-isometry, in particular under passing to finite-index subgroups.

The asymptotic behaviour of the growth function has been studied since at least the 1960s by Milnor, Wolf, Bass, Guivarc'h and others. One of the highlights is Gromov's celebrated result [37] that the groups with polynomial growth are precisely those that are virtually nilpotent, cementing the link between geometry and algebraic structure. The growth function may also be exponential (for example the free group), and it remained an open question whether their existed groups of intermediate growth (strictly between polynomial and exponential) until such a group was discovered by Grigorchuk [36] in 1980.

We can also consider the *generating function* associated to β, which is the formal power series $S(z) = \sum_{n \geq 0} \beta(n) z^n$. This object is more algebraic in nature and therefore different questions are appropriate. We are interested in the algebraic complexity, that is, we look to categorise when $S(z) \in \mathbb{Q}(z)$, or more generally when it is algebraic over $\mathbb{Q}(z)$, the field consisting of rational functions of the form $\frac{p(z)}{q(z)}$ where p and q are polynomials with rational coefficients. One implication of having rational growth series is that Dehn's word problem has a solution. That is, there exists an algorithm that decides whether or not a given word in S^* represents the group identity.

Algebraic complexity is not as thoroughly studied as asymptotics. One of the difficulties is that unlike the asymptotic behaviour, the algebraic complexity may depend on the presentation. Stoll [57] has shown that there exist nilpotent groups possessing both rational and transcendental growth series, depending on the choice of generating set.

However, in certain cases the complexity is known to be the same for all generating sets. This can be interpreted as arising from some underlying structure in the group, rather than just an artefact of the presentation. In 1983, Benson [4] proved that every virtually abelian group has rational growth series, with respect to any finite generating set. Around the same time, hyperbolic groups were shown to have the same property, with credit due to Cannon, Thurston, and Gromov [9] [8] [25] [38]. Much more recently, Duchin and Shapiro [24] have proved that the integer Heisenberg group (the free nilpotent group of class 2 on 2 generators) has rational growth for all generating sets. To the author's knowledge, there are no other results on (standard) growth series which hold regardless of generating set. A number of other groups are known to have rational growth for some choice of generating set, for example the soluble Baumslag-Solitar groups [17], so-called ‘higher Baumslag-Solitar’ groups [55], certain automatic groups, Coxeter groups, some Artin-Tits groups.

One way to view standard growth is to consider the equivalence relation on S^* where words are equivalent whenever they represent the same group element. We then choose one minimal length word from each equivalence class, and study the growth of the resulting language. Any other equivalence relation on S^* yields a growth function (and corresponding series) in the same way. With a view to algebraic relevance, we consider the equivalence relation where words are equivalent whenever they represent elements in the same coset, for some fixed subgroup, and the relation where words are equivalent whenever they represent elements of the same conjugacy class. The corresponding growth functions/series are known as *coset growth* and *conjugacy growth* (denoted $C_G(n)$) respectively.

We can also restrict ourselves to a subset of the group, and consider the growth of just the elements (or cosets or conjugacy classes) in this subset, with respect to the metric inherited from the main group. We refer to this as *relative growth*. This thesis deals mostly with conjugacy growth, but also with coset growth and relative growth.

Conjugacy growth was introduced explicitly by Babenko in [2] in the guise of counting geodesics on Riemannian manifolds, although Margulis [48] had studied similar counting problems prior to that. The asymptotics have also been studied by Coornaert and Knieper [18], Breuillard, Cornulier, Lubotzy and Meiri [6], [5], Guba and Sapir [40], and Hull and Osin [45].

Turning to the formal power series, Ciobanu and Hermiller [12] used formal language approaches to study the conjugacy growth series of free and direct products. Ciobanu, Hermiller, Holt, and Rees [13] proved that virtually cyclic groups always have rational conjugacy growth. Antolín and Ciobanu [1] proved that non-elementary hyperbolic groups (those which are neither finite nor virtually cyclic) always have transcendental conjugacy growth. This verifies a conjecture of Rivin [53], [54]. More recently, Gekhtmann and Yang [33] have shown that all relatively hyperbolic groups, and some acyclindrically hyperbolic groups, have transcendental conjugacy growth, again with respect to all generating sets. Mercier [49] has shown that a number of wreath products have transcendental conjugacy growth, with respect to certain generating sets.

For a useful, if not so recent, survey on the algebraic complexity of various notions of growth, including those described here, see Gromov and De La Harpe's article [35].

We now summarise the content of the rest of the thesis.

In **Chapter 2**, we introduce the basic definitions and results that we will need.

Chapter 3 is devoted to virtually abelian groups, and based on the author's paper [26]. We use the notions of patterns and polyhedral sets, developed by Benson [4], to study various growth series. A key step here is assigning positive integer weights to the generators, and working with the *weighted length* of elements, the minimal weight of words representing the element. The more familiar case where the weights are uniformly 1 is a special case of these stronger statements.

Significantly, the results are independent of the choice of finite generating set, and of the choice of weight function. Let G be a virtually abelian group, with any choice of finite weighted generating set S, and let H be an arbitrary subgroup. We prove the following.

Theorem 3.2.1. *The set of right cosets $H\backslash G$ has rational weighted growth series.*

Theorem 3.3.1. *The group G has rational weighted conjugacy growth series.*

Theorem 3.4.2. *The subgroup H has rational weighted relative growth series.*

Chapter 4 deals with certain nilpotent groups of class 2. We draw together and extend work of Babenko [2] and Stoll [57], making asymptotic estimates of the conjugacy growth for these groups.

Stoll classifies the class 2 nilpotent groups with infinite cyclic derived subgroup, using a generalised version of the direct product, known as a *centrally amalgamated direct product*. Every such group can be constructed as a quotient of the direct product of a finite number of copies of the integer Heisenberg group H_1 (the free nilpotent group of class 2 on 2 generators). The number of copies of H_1 is called the *Heisenberg rank*, r, of the group. We prove the following generalisation of a result of Babenko, and its implication for the conjugacy growth series.

Theorem 4.3.3. *Let G be a class 2 nilpotent group, with infinite cyclic derived* *subgroup, and let its Heisenberg rank be* $r \geq 1$. *Then there exists* $s \in \mathbb{N}$ *such that*

$$c_G(n) \sim \begin{cases} n^{2+s} \log n & r = 1 \\ n^{2r+s} & r \geq 2. \end{cases}$$

Corollary 4.3.4. *If* G *is a class 2 nilpotent group with infinite cyclic derived subgroup, with Heisenberg rank equal to 1, then the conjugacy growth series of* G *is transcendental (with respect to any finite generating set).*

Babenko also proves the following Theorem.

Theorem 4.5.1. *If* G *has a finite index subgroup isomorphic to the Heisenberg group, then* G *has cumulative conjugacy growth function equivalent to* $n^2 \log n$.

We provide a proof that broadly follows the ideas given by Babenko but includes full details. We also have the following implication for the conjugacy growth series.

Corollary 4.5.3. *If* G *has a finite-index subgroup isomorphic to the Heisenberg group, then* G *has transcendental conjugacy growth (with respect to any finite generating set).*

We finish Chapter 4 with a brief discussion of free nilpotent groups of class 2, and provide a counter-example to an unproved claim of Guba and Sapir [40].

Chapter 5 comprises joint work [11] with Laura Ciobanu and Turbo Ho on the soluble Baumslag-Solitar groups

$$BS(1,k) = \langle a, t \mid tat^{-1} = a^k \rangle.$$

Collins, Edjvet, and Gill [17] have proved that the standard growth series for these groups is rational with respect to the generating set $\{a, t\}$. Building on this work, we use a mixture of formal language theory and combinatorics to show the following.

Theorem (Corollary 5.3.2)**.** The groups $BS(1,k)$ have transcendental conjugacy growth series with respect to the generating set $\{a, t\}$.

Moreover, we produce precise formulae for the growth series. From these we obtain the conjugacy growth rate (see Section 2.2.1), and compare with results of Bucher and Talambutsa [7] to prove the following.

Corollary 5.3.3. *The conjugacy and standard growth rates of $BS(1,k)$, with respect to the generating set $\{a,t\}$, are equal.*

Chapter 6 contains conjectures, open questions, and suggestions for future work.

Chapter 2

Preliminaries

We record some basic results that form the background to this work.

2.1 Notation and basic results

We fix some notational conventions. In this thesis, $\mathbb{N}$ will contain zero, and we will write $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. If U is a set, we write $\#U$ for the cardinality of U.

For an alphabet S (i.e. a finite collection of symbols), we write S^* to denote all words over S, that is the set $\{s_1 s_2 \cdots s_k \mid s_i \in S, k \in \mathbb{N}\}$. We denote the empty word by ϵ. If $w = s_1 s_2 \cdots s_k \in S^*$, we write $|w| = k$ for the length of the word w.

Suppose we have a group presentation $G = \langle S \mid R \rangle$, and $w \in S^*$ is a word over the generators. We write $\overline{w}$ to denote the group element represented by w.

We will use the following standard notation. Write $f(n) = \mathcal{O}(g(n))$ if there is some $C > 0$ for which $f(n) \leq Cg(n)$ for all large enough n, and $f(n) = o(g(n))$ if $\frac{g(n)}{f(n)} \to 0$ as $n \to \infty$.

All groups considered in this thesis will be finitely generated.

The following is standard (see, for example, Mann [47]).

Lemma 2.1.1. *Suppose G is finitely generated and contains a finite-index subgroup H. Then G has a* characteristic *subgroup N contained in H with $[G\colon N]$ finite.*

Proof. Let

$$N = \bigcap_{[G\colon H'] = [G\colon H]} H'.$$

Clearly this is contained in H. For subgroups $X, Y \leq G$, we have $(X \cap Y)g = (Xg) \cap (Yg)$. The number of distinct subsets of the form $(Xg) \cap (Yg)$ is at most $[G\colon X][G\colon Y]$, and so $[G\colon X \cap Y] \leq [G\colon X][G\colon Y]$. Therefore $[G\colon N]$ is finite, as finite generation implies there are only finitely many subgroups of index $[G\colon H]$ in G.

Suppose $\phi \in \mathrm{Aut}G$. Then ϕ maps cosets of H to cosets of $\phi(H)$. It is easy to see that this is a bijective map, and thus $[G\colon \phi(H)] = [G\colon H]$. Now since ϕ is an automorphism, it defines a permutation of the set of subgroups of fixed finite index. Thus if $h \in N$, $\phi(h) \in N$. So N is characteristic, and the lemma is proved. □

2.2 Growth

This section is standard. See, for example, Mann [47] or de la Harpe [19].

Definition 2.2.1. Let G be a group with finite generating set S. Then the *word length* of an element $g \in G$ is defined as

$$|g|_S = \min\{|w| \mid w \in S^*,\ \overline{w} = g\}.$$

Definition 2.2.2. For a group G with finite generating set S, define the strict and cumulative growth functions respectively as

- $\sigma_{G,S}(n) = \#\{g \in G \mid |g|_S = n\}$
- $\beta_{G,S}(n) = \#\{g \in G \mid |g|_S \leq n\}$

Definition 2.2.3. Let $f, g\colon \mathbb{N} \to \mathbb{N}$ be two functions. We write $f \preccurlyeq g$ if there exists $\lambda > 1$ such that

$$f(n) \leq \lambda g(\lambda n) + \lambda$$

for all $n \in \mathbb{N}$. If $f \preccurlyeq g$ and $g \preccurlyeq f$ then we write $f \sim g$, and say that the functions are *equivalent.*

Write $f \prec g$ if $f \preccurlyeq g$ but $f \nsim g$.

Proposition 2.2.4. *Let S and T be two finite generating sets for a group G. Then $\beta_{G,S} \sim \beta_{G,T}$.*

Thus we may refer to the growth type of a group without ambiguity, and we have the following obvious trichotomy.

Theorem 2.2.5. *For a group G we have exactly one of the following:*

1. $\beta_G(n) \prec n^d$ *for some* $d \in \mathbb{N}$,
2. $\beta_G(n) \succ a^n$ *for some* $a > 1$, *or*
3. $\beta_G(n) \prec a^n$ *for all* $a > 1$ *and* $\beta_G(n) \succ n^d$ *for all* $d \in \mathbb{N}$

We say that G has polynomial, exponential, *or* intermediate *growth, in the situations above.*

Theorem 2.2.6. *Let G be some finitely generated group and H a finite-index subgroup. Then their growth functions are equivalent. That is, we have $\sigma_G \sim \sigma_H$ and $\beta_G \sim \beta_H$.*

Proposition 2.2.7. *The free abelian group $\mathbb{Z}^s$ has cumulative growth function equivalent to n^s.*

Definition 2.2.8. Let $U \subseteq G$ be any subset of G. We define the *relative* cumulative and strict growth of U as follows:

- $\beta_{U \mathrm{rel} G,S}(n) = \#\{g \in G \mid |g|_S \leq n, g \in U\}$
- $\sigma_{U \mathrm{rel} G,S}(n) = \#\{g \in G \mid |g|_S = n, g \in U\}$.

Definition 2.2.9. Let H be a subgroup of G. Define the length of a coset Hg as $|Hg|_S = \min\{|\gamma|_S \mid \gamma \in Hg\}$. Then we define the corresponding (right) coset growth as follows:

- $\beta_{H\backslash G,S}(n) = \#\{Hg \in H\backslash G \mid |Hg|_S \leq n\}$
- $\sigma_{H\backslash G,S}(n) = \#\{Hg \in H\backslash G \mid |Hg|_S = n\}$.

One could of course consider the left coset growth symmetrically.

Definition 2.2.10. Let $\mathcal{C}(G)$ denote the set of conjugacy classes of G. Then the length of a conjugacy class $\kappa \in \mathcal{C}$ is defined as $|\kappa|_S = \min\{|g|_S \mid g \in \kappa\}$. Then we define the cumulative and strict conjugacy growth of G as follows:

- $c_{G,S}(n) = \#\{\kappa \in \mathcal{C} \mid |\kappa|_S \leq n\}$
- $s_{G,S}(n) = \#\{\kappa \in \mathcal{C} \mid |\kappa|_S = n\}$.

With all of the above growth functions, we may sometimes omit the generating set, or the group, from the notation when it will not cause confusion.

In the following lemma, we see that the relative growth of a finite-index subgroup is equivalent to that of its cosets.

Lemma 2.2.11. *Let G be a f.g. group with a finite-index subgroup Γ, and choose a transversal T. Then there exists a constant k such that for any $t \in T$ and $h \in \Gamma$ we have $|h| - k \leq |th| \leq |h| + k$ (where the length is the length in G).*

Proof. By the triangle inequality we have $|th| \leq |t| + |h|$. And since $h = t^{-1}th$, we also have $|h| \leq |t^{-1}| + |th|$. Thus $|h| - |t^{-1}| \leq |th| \leq |h| + |t|$. Let $k = \max\{|u| \mid u \in T \cup T^{-1}\}$. □

2.2.1 Growth Rate

Definition 2.2.12. Let $(a_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence of positive integers. Then the *growth rate* of a_n is defined as

$$\rho = \limsup_{n\to\infty} \sqrt[n]{a_n}.$$

If $\beta(n)$ is the cumulative standard growth function of a finitely generated group, then the lim sup is in fact a limit. Write ρ_S where S is the finite generating set in question.

Remark 2.2.13. *The reciprocal of ρ, that is $\frac{1}{\rho}$, is the radius of convergence of the generating function $\sum_{n\geq 0} \beta(n)z^n$.*

Definition 2.2.14. If $\inf_S\{\rho_S \mid S \text{ is a finite generating set of } G\} > 1$, then G is said to have *uniform exponential growth.*

Uniform exponential growth is a popular topic of research, see for example de la Harpe [20], and more recently Kar and Sageev [46]. Breuillard, Cornulier, Lubotzky, and Meiri have shown [5] that a linear group which is not virtually nilpotent has uniform exponential *conjugacy* growth.

2.3 Generating Functions

We write $\mathbb{Q}[[z]]$ for the ring of formal power series over a variable z with coefficients in $\mathbb{Q}$, and $\mathbb{Q}[z]$ for the ring of polynomials over z with rational coefficients.

Definition 2.3.1. 1. An element $\Gamma(z) \in \mathbb{Q}[[z]]$ is said to be *rational* if it is an element of the field of fractions of $\mathbb{Q}[z]$, denoted $\mathbb{Q}(z)$. In other words, there are polynomials $p, q \in \mathbb{Q}[z]$ such that $\Gamma = \frac{p}{q}$.

2. An elements $\Gamma(z) \in \mathbb{Q}[[z]]$ is said to be *algebraic* if it is algebraic over $\mathbb{Q}[z]$. In other words, it is the root of some polynomial expression with coefficient from the ring of polynomials $\mathbb{Q}[z]$.

3. An element $\Gamma(z) \in \mathbb{Q}[[z]]$ is said to be *transcendental* if it is not algebraic.

We will refer to this classification as the *algebraic complexity* of $\Gamma(z)$.

Definition 2.3.2. Let $\gamma(n)$ be any of the growth functions defined above. Then we define the associated *growth series* of γ to be:

$$\Gamma(z) = \sum_{n=0}^{\infty} \gamma(n) z^n \in \mathbb{Q}[[z]].$$

We shall study the algebraic complexity of the various growth series for different classes of finitely generated groups.

The following lemma shows that the distinction between strict and cumulative growth is unimportant when we are considering algebraic complexity.

Lemma 2.3.3. *Let $\gamma(n)$ denote a strict growth function, and $\delta(n)$ its cumulative counterpart (i.e. $\delta(n) = \sum_{i=0}^{n} \gamma(i)$). Then the algebraic complexity of $\sum_{n\geq 0} \gamma(n) z^n$ is the same as that of $\sum_{n\geq 0} \delta(n) z^n$.*

Proof. The standard formula for the product of generating functions is

$$\sum_{n=0}^{\infty} f(n) z^n \sum_{n=0}^{\infty} g(n) z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} f(k) g(n-k) \right) z^n.$$

Letting $f(n) = 1$ for all n we get

$$\frac{1}{1-z} \sum_{n=0}^{\infty} g(n) z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} g(n-k) \right) z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} g(k) \right) z^n.$$

Now since $\delta(n) = \sum_{k=0}^{n} \gamma(n)$, we have

$$\sum_{n=0}^{\infty} \delta(n) z^n = \frac{1}{1-z} \sum_{n=0}^{\infty} \gamma(n) z^n.$$

So $\sum \delta(n) z^n$ is rational/algebraic/transcendental if and only if $\sum \gamma(n) z^n$ is rational/algebraic/transcendental. □

Generating functions are a well-studied topic. Duchin has written a very readable introduction [23] to generating functions and growth. We use the rigorous treatments in [34] and [56].

The connections between the asymptotic behaviour and the algebraic complexity are somewhat subtle. For example, the following is an immediate consequence of Theorem 4.1.1 of [56].

Proposition 2.3.4. *If $\sum \gamma(n) z^n$ is rational then $\gamma(n)$ has either polynomial or exponential growth.*

The next result relates to generating functions whose coefficients are in the polynomial range. We first need a definition:

Definition 2.3.5. A function $f : \mathbb{N} \to \mathbb{C}$ is called *eventually quasi-polynomial* if there exist some positive integer period N, threshold $T \geq 0$, and polynomials $f_0, f_1, \ldots, f_{N-1}$ so that for all $n \geq T$, $f(n) = f_i(n)$ whenever $n \equiv i \mod N$.

The following is an immediate consequence of Proposition 4.4.1 of [56].

Proposition 2.3.6. *Let $\gamma(n) \leq Cn^d$ for some $C > 1$, $d \in \mathbb{N}$. Then $\sum_{n \geq 0} \gamma(n) z^n$ is rational if and only if $\gamma(n)$ is eventually quasi-polynomial.*

This has the following application that we will use in Chapter 4.

Corollary 2.3.7. *Suppose $\gamma : \mathbb{N} \to \mathbb{N}$ is non-decreasing and bounded by a polynomial as above. If $\sum_{n \geq 0} \gamma(n) z^n$ is rational then $\gamma \sim n^d$ for some $d \in \mathbb{N}$.*

Proof. Proposition 2.3.6 implies that γ is eventually quasi-polynomial, say with polynomials $\gamma_0, \ldots, \gamma_N$ as in the definition. The degree of each γ_i is at least the degree of γ_{i-1} (since large enough n would violate the non-decreasing assumption). But since these polynomials cycle, the degree of γ_0 must be at least the degree of

γ_N. So they all have the same degree, say d. So γ cycles between finitely many polynomials, all equivalent to n^d, and so $\gamma(n) \sim n^d$. $\square$

We list two more results on series that we will need.

Proposition 2.3.8 (Stoll, Proposition 3.3 of [57]). *Let* $\Gamma(z) = \sum_{n\geq 0} \gamma(n) z^n \in \mathbb{Q}[[z]]$*, and suppose that* $\lim_{n\to\infty} \frac{\gamma(n)}{n^d} = a$*. Then if* a *is an irrational (resp. transcendental) number then* $\Gamma(z)$ *is irrational (resp. transcendental) as a series.*

Theorem 2.3.9 (Fatou [28]). *A series* $\sum a_n z^n \in \mathbb{Z}[[z]]$ *that converges inside the unit disk is either rational or transcendental.*

2.4 General results on conjugacy growth

We now detail some essential background results regarding conjugacy growth. Firstly, conjugacy growth type is a group property.

Proposition 2.4.1. *Let* S *and* T *be two finite generating sets of a group* G*. Then* $c_{G,S} \sim c_{G,T}$*.*

Proof. Let $\Lambda_1 = \max\{|s|_T \mid s \in S\}$ and $\Lambda_2 = \max\{|t|_S \mid t \in T\}$. Suppose $[g]$ is a conjugacy class of G with $||[g]||_S = n$. That is, there is some $\gamma \in G$ and $s_1, s_2, \ldots, s_n \in S$ with $\gamma g \gamma^{-1} = s_1 s_2 \cdots s_n$. Then $|\gamma g \gamma^{-1}|_T \leq \Lambda_1 n$ and so $||[g]||_T \leq \Lambda_1 ||[g]||_S$. A symmetrical argument gives $||[g]||_S \leq \Lambda_2 ||[g]||_T$. Let $\Lambda = \max\{\Lambda_1, \Lambda_2\}$. Now if $||[g]||_S \leq n$ then $||[g]||_T \leq \Lambda n$, and so $c_{G,S}(n) \leq c_{G,T}(\Lambda n)$. Symmetrically, $c_{G,T}(n) \leq (\Lambda n)$ and so we have $c_{G,T}(\frac{1}{\Lambda} n) \leq c_{G,S}(n) \leq c_{G,T}(\Lambda n)$. $\square$

On the other hand, conjugacy growth is not a geometric property. That is to say, passing to a finite-index subgroup may not preserve the asymptotic behaviour, unlike standard growth. Hull and Osin [45] exhibit a finitely generated group with exponential conjugacy growth possessing a finite-index subgroup with only two conjugacy classes (and hence the minimum possible conjugacy growth for a non-trivial group).

The following is stated without proof in [47] (Proposition 17.11).

Proposition 2.4.2. *Let G be generated by S and H be generated by T. Then the direct product $G \times H$ is generated by $S \cup T$ and their spherical conjugacy growth series satisfy the following:*

$$\sum_{n=0}^{\infty} s_{G\times H,S\cup T}(n)z^n = \sum_{n=0}^{\infty} s_{G,S}(n)z^n \cdot \sum_{n=0}^{\infty} s_{H,T}(n)z^n$$

.

Proof. Let $g \in G$ and $h \in H$ and suppose $|[g]|_S = n$ and $|[h]|_T = m$. So their exist elements $\gamma \in G$ and $\delta \in H$, and generators $s_1, \dots s_n \in S$, $t_1, \dots, t_m \in T$, so that $\gamma g \gamma^{-1} = s_1 \cdots s_n$ and $\delta h \delta^{-1} = t_1 \cdots t_m$. In the direct product we then have

$$\gamma g \gamma^{-1} \delta h \delta^{-1} = \gamma \delta g h \delta^{-1} \gamma^{-1}$$

and so $|[gh]|_{S\cup T} \leq n + m = |[g]|_S + |[h]|_T$.

On the other hand, let $gh \in G \times H$ and suppose $|[gh]|_{SUT} = l$. So there exists an element $xy \in G \times H$ with $xyghy^{-1}x^{-1} = u_1 u_2 \cdots u_l$. Rearranging, we have $xgx^{-1} \cdot yhy^{-1} = s_1 \cdots s_n t_1 \cdots t_m$ for some $n, m \in \mathbb{N}$ with $l = n + m$. Thus $xgx^{-1} = s_1 \dots s_n$ and $yhy^{-1} = t_1 \cdots t_m$ so $|[g]|_S \leq n$ and $|[h]|_T \leq m$, and we have $|[g]|_S + |[h]|_T \leq |[gh]|_{S\cup T}$. So $|[gh]|_{S\cup T} = |[g]|_S + |[h]|_T$.

Now if $|[gh]|_{S\cup T} = n$ then there exists $0 \leq i \leq n$ with $|[g]|_S = i$ and $|[h]|_T = n - i$; and conversely if $|[g]|_S = l$ and $|[h]|_T = m$ then $|[gh]|_{S\cup T} = l + m$. So $s_{G\times H,S\cup T}(n)$ is precisely the number of pairs of conjugacy classes whose lengths sum to n, thus

$$s_{G\times H}(n) = \sum_{i=0}^{n} s_{G,S}(i)s_{H,T}(n-i).$$

Finally, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} s_{G\times H,S\cup T}(n)z^n &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} s_{G,S}(i)s_{H,T}(n-i) \right) z^n \\
&= \sum_{n=0}^{\infty} s_{G,S}(n)z^n \cdot \sum_{n=0}^{\infty} s_{H,S}(n)z^n.
\end{aligned}$$

$\square$

Turning to cumulative growth, we have the following.

Lemma 2.4.3. *Let G and H be finitely generated groups. Then $c_{G\times H} \sim c_G \cdot c_H$.*

Proof. First, note that for $(g,h) \in G \times H$ we have

$$[(g,h)] = \{(g,h)^{(x,y)} \mid (x,y) \in G \times H\} = \{(g^x, h^y) \mid (x,y) \in G \times H\} = [g] \times [h].$$

Now if $|[(g,h)]|_{S\cup T} \leq n$ then there exists $(x,y) \in G \times H$ with $(x,y)(g,h)(x,y)^{-1} = u_1 u_2 \cdots u_l$ for some $u_i \in S \cup T$, $l \leq n$. Rearranging, we find elements $s_1, \ldots, s_{l_1} \in S$ and $t_1, \ldots, t_{l_2} \in T$, with $l_1 + l_2 = l$, so that

$$(xgx^{-1}, ygy^{-1}) = s_1 \cdots s_{l_1} t_1 \cdots t_{l_2}.$$

Therefore $|xgx^{-1}|_S \leq l_1 \leq n$ and $|ygy^{-1}|_T \leq l_2 \leq n$, and so $|[g]|_S \leq n$ and $|[h]|_T \leq n$. Thus $c_{G\times H}(n) \leq c_G(n) \cdot c_H(n)$.

Conversely, suppose $|[g]|_S \leq n$ and $|[h]|_S \leq n$. Then there are elements $\gamma \in G$ and $\delta \in H$ such that $|\gamma g \gamma^{-1}|_S \leq n$ and $|\delta h \delta^{-1}|_T \leq n$, so $|(\gamma g \gamma^{-1}, \delta h \delta^{-1})|_{S\cup T} \leq 2n$. But $(\gamma g \gamma^{-1}, \delta h \delta^{-1}) = (\gamma, \delta)(g,h)(\gamma,\delta)^{-1}$ and so $|[(g,h)]|_{S\cup T} \leq 2n$. Thus $c_G(n) \cdot c_H(n) \leq c_{G\times H}(2n)$, giving the required result. □

The following two lemmas appear in [6], the first without proof.

Lemma 2.4.4 (Lemma 3.1 (1) of [6])**.** *Let H be a quotient of G. Then $c_H(n) \preccurlyeq c_G(n)$.*

Proof. Let $\theta\colon G \to H$ denote the natural homomorphism. Choose a finite, inverse-closed, generating set S for G which contains the identity. Then $\theta(S)$ is a finite generating set for H. Suppose $h \in H$ has length at most n with respect to $\theta(S)$. Then there exist $\sigma_1, \ldots, \sigma_n \in S$ such that $h = \theta(\sigma_1) \cdots \theta(\sigma_n)$. Then $g := \sigma_1 \cdots \sigma_n \in G$ is a lift of h, with $|g|_S \leq n$.

Suppose g_1 and g_2 are conjugate elements of G, say $g_1 = \gamma g_2 \gamma^{-1}$ for some $\gamma \in G$. Then $\theta(g_1) = \theta(\gamma)\theta(g_2)\theta(\gamma)^{-1} \in H$. So if h_1, h_2 are non-conjugate in H, any pair of lifts in G must be non-conjugate.

Since there are $c_{H,\theta(S)}(n)$ non-conjugate elements in H, of length at most n, there must be at least this number of non-conjugate elements in G, with lengths at most n, i.e. $c_{H,\theta(S)}(n) \leq c_{G,S}(n)$. □

Lemma 2.4.5 (Lemma 3.1 (2) of [6]). *Let H be a finite-index subgroup of G. Then $c_H \preccurlyeq c_G$.*

Proof. Let $k = [G : H]$, and suppose that $S \subset G$ is a finite symmetric generating set. It is standard (see for example [47, Chapter 2]) that there exists a subset $S' \subset G$ that is a finite symmetric generating set for H, all of whose elements are of length at most $2k-1$. Thus $B_{H,S'}\left(\frac{n}{2k-1}\right) \subset B_{G,S}(n)$. So by definition of $c_{H,S'}$, $B_{G,S}$ contains a subset Y of size $c_{H,S'}\left(\frac{n}{2k-1}\right)$, consisting of pairwise non-H-conjugate elements. Now passing to G-conjugation can only reduce the number of pairwise non-conjugate elements by a factor of k. So Y contains at least $\frac{1}{k}c_{H,S'}\left(\frac{n}{2k-1}\right)$ pairwise non-G-conjugate elements. Thus there are at least this many pairwise non-G-conjugate elements in $B_{G,S}(n)$, i.e. $\frac{1}{k}c_{H,S'}\left(\frac{n}{2k-1}\right) \leq c_{G,S}(n)$, which gives the result. □

2.5 Formal Language Theory

One of the standard ways to study the algebraic complexity of growth series is via formal language theory. We will need the following background in Chapter 5. A standard source is [44].

Definition 2.5.1. Let $\mathcal{V}$ be a set of variables (usually denoted by upper case letters), and $\mathcal{T}$ a set of terminals (usually denoted by lower case letters). A *context-free grammar* consists of a finite set of production rules of the form

$$V \to w_1 \mid w_2 \mid \cdots \mid w_n$$

where $V \in \mathcal{V}$, each $w_i \in (\mathcal{V} \cup \mathcal{T})^*$, and the $\mid$ symbol stands for exclusive 'or'. We nominate one variable to be the starting variable.

A context-free grammar produces a language in the following way. Start at the nominated starting variable, and perform substitutions according to the production rules, until the word consists only of terminals. The language $L \subset \mathcal{T}^*$ of all words that can be produced from the grammar is called a *context-free* language. If each word is only produced in one way (i.e. via a unique sequence of production rules) then the language is called *unambiguous context-free*.

Theorem 2.5.2 (Chomsky-Schützenberger [10]). *If L is an unambiguous context-free language, its growth series is algebraic.*

There is a method for explicitly calculating the series, known as the DSV method, which is as follows. A useful exposition can be found in [15]. Convert the grammar into a system of equations by replacing:

- the empty word ϵ with the integer 1,
- each terminal with the formal variable z,
- each variable V with a function $V(z)$,
- the or $|$ with addition $+$,
- concatenation with multiplication,
- the production arrow with $=$.

Solving the system of equations for the initial variable then gives the growth series, an algebraic function of z.

Remark 2.5.3. *There is a subclass of the context-free languages called* regular *languages, whose growth series are always rational. Many of the existing rationality results in the growth of groups rely in regular languages. However we will not use them in this thesis.*

Chapter 3

Virtually Abelian Groups

This chapter is based on the author's paper [26]. In it, we focus on the formal power series associated to various growth functions in virtually abelian groups. Benson [4] proved that the standard growth series is rational with respect to all generating sets. We generalise and extend his arguments to show that the power series associated to coset growth, conjugacy growth, and relative growth of certain subsets are all rational (see Theorem 3.2.1, Theorem 3.3.1, and Theorem 3.4.1 respectively). The author wishes to thank the anonymous referee's comments regarding presentation and clarity of the vectors and matrices within this chapter.

In this chapter, we will always be considering a (finitely generated) virtually abelian group G. Any abelian subgroup of finite index will also be finitely generated, so it has the form $\mathbb{Z}^r \times T$ for some $r \in \mathbb{N}$, and finite abelian T. So G is also virtually $\mathbb{Z}^r$. By Lemma 2.1.1, there is a normal, finite-index subgroup of G contained in $\mathbb{Z}^r$ (which is therefore free abelian). So from now on we will assume that G contains $\mathbb{Z}^n$ as a finite-index normal subgroup, for some $n \in \mathbb{N}_+$. Unless otherwise stated, we will let $d = [G \colon \mathbb{Z}^n]$. Finally, we will write $X^\top$ for the transpose of a vector or matrix X.

The asymptotic behaviour of conjugacy growth is well-understood, as we can see in the following Proposition.

Proposition 3.0.1. *The (cumulative) standard and conjugacy growth functions of a virtually abelian group G are equivalent.*

Proof. By Theorem 2.2.6 and Proposition 2.2.7, we have $\beta_G(n) \sim n^d$, and so $c_G(n) \preccurlyeq n^d$ (since conjugacy growth is clearly bounded above by standard growth). On the other hand, Lemma 2.4.4 gives $c_G(n) \succcurlyeq c_H(n) \sim \beta_H(n) \sim n^d$ (since a conjugacy class in an abelian group is simply an element). Therefore $c_G(n) \sim n^d \sim \beta_G(n)$. $\square$

3.1 Definitions and basic results

3.1.1 Patterns

For the remainder of the chapter we work with strict growth functions. Here we follow [4]. Choose a finite monoid generating set Σ. A function $\omega\colon \Sigma \to \mathbb{N}_+$ will be called a *weight function*. We extend this to $\omega\colon \Sigma^* \to \mathbb{N}_+$, so that $\omega(s_1 s_2 \cdots s_l) = \omega(s_1) + \omega(s_2) + \cdots + \omega(s_l)$ for any word $s_1 s_2 \cdots s_l$. Define the weight of a group element as

$$\omega(g) = \min\{\omega(w) \mid w \in \Sigma^*,\ \overline{w} = g\}.$$

If $\omega(s) = 1$ for all $s \in \Sigma$, this gives the usual notion of word length.

Now consider an extended generating set $S = S_\Sigma$.

Definition 3.1.1. With Σ as above, let

$$S = \{s_1 s_2 \cdots s_k \mid s_i \in \Sigma,\ 1 \leq k \leq d\}. \tag{3.1}$$

The weight of a generator $s_1 s_2 \cdots s_k \in S$ is defined with respect to our original weight function: $\omega(s_1 s_2 \cdots s_k) = \sum_i \omega(s_i)$. Thus the weight of an element with respect to the new weighted generating set S is equal to its weight with respect to Σ, and so the respective weighted growth series are equal.

We need the following important observation.

Lemma 3.1.2. *Any product $g_1 g_2 \cdots g_k$ of elements in G, where $k \geq d$, must contain a subproduct $g_{e+1} \cdots g_f \in \mathbb{Z}^n$.*

Proof. Consider the products g_1, $g_1 g_2$, up to $g_1 g_2 \cdots g_k$. There are k of these, and there are d cosets in $G/\mathbb{Z}^n$, so if $k > d$, some pair of these products must be in the same coset, say $g_1 g_2 \cdots g_i$ and $g_1 g_2 \cdots g_j$ with $j > i$. That is $g_1 g_2 \cdots g_i \mathbb{Z}^n = g_1 g_2 \cdots g_j \mathbb{Z}^n$, and so $g_{i+1} g_{i+2} \cdots g_j \in \mathbb{Z}^n$. If $k = d$, and each subproduct is in a distinct coset, one of the subproducts must be in $\mathbb{Z}^n$, the identity coset. $\square$

Definition 3.1.3. Write $X := S \cap \mathbb{Z}^n = \{x_1, \ldots, x_r\}$ and $Y := S \setminus (S \cap \mathbb{Z}^n) = \{y_1, \ldots, y_s\}$, and call any word in Y^* a *pattern*.

Note that by Lemma 3.1.2, any element of S of the form $s_1 \cdots s_d$ contains a subword $s_t s_{t+1} \cdots s_u \in \mathbb{Z}^n$. And since $s_t s_{t+1} \cdots s_u \in S$ by definition, X is always non-empty.

Definition 3.1.4. Let $\mathrm{patt}\colon S \to Y$ be the map

$$\mathrm{patt}\colon s_i \mapsto \begin{cases} \epsilon & \text{if } s_i \in X \\ s_i & \text{if } s_i \in Y \end{cases}.$$

This extends to a monoid homomorphism $\mathrm{patt}\colon S^* \to Y^*$, which records those generators in a word which are not contained in $\mathbb{Z}^n$. For $w \in S^*$ we call $\mathrm{patt}(w)$ the *pattern* of w.

There are of course infinitely many patterns in Y^*. We now prove that, with respect to the generating set S, we only need finitely many to describe minimal-weight representatives for all elements of G.

Proposition 3.1.5 (Proposition 11.3 of [4])**.** *Suppose $g \in G$. Then there exists a minimal weight word $w \in S^*$, with a pattern of length at most d, such that $\overline{w} = g$.*

To prove the proposition, we introduce a new property of words. Let L be a positive integer. Then for a word $w \in S^*$, define the height, μ_L, to be the sum of the length of its pattern and the weight of its pattern multiplied by L:

$$\mu_L(w) = \omega(\mathrm{patt}(w))L + |\mathrm{patt}(w)|_S \,.$$

We say a word in S^* has *minimal height* if it has minimal μ_L over all words that represent the same element. Next, we show that if a minimal weight word contains an adjacent pair of elements of Y, then it cannot have minimal height.

Lemma 3.1.6. *Suppose s and t are elements of Y, and st is a minimal weight word representing an element $g \in G$. Then there exists $u \in S^*$ that represents g with $\omega(u) = \omega(g)$, such that $\mu_L(st) > \mu_L(u)$, for all $L \geq |g|_\Sigma$.*

Proof. We have $\mu_L(st) = \omega(\text{patt}(st))L + |\text{patt}(st)|_S = \omega(st)L + |st|_S = \omega(g)L + 2$. Let $s_1 s_2 \cdots s_j$ and $s_{j+1}s_{j+2}\cdots s_k$ be minimal weight words in Σ^* that represent s and t respectively.

Suppose that $k < d$. Then $u := s_1 \cdots s_j s_{j+1} \cdots s_k \in S$, and is a minimal weight representative for g. It may be the case that $u \in \mathbb{Z}^n$. If so, $\text{patt}(u)$ is the empty word and so $\mu_L(u) = 0 < \omega(g)L + 2 = \mu_L(st)$. If not, then $\text{patt}(u) = u$ and we have $\mu_L(u) = \omega(u)L + 1 = \omega(g)L + 1 < \omega(g)L + 2$. So in both cases u is a minimal weight word representing g that has strictly smaller height than st.

Now suppose $k \geq d$. By Lemma 3.1.2, there is some subproduct of length at most d, say $s_{e+1}s_{e+2}\cdots s_f$, that represents an element of $\mathbb{Z}^n$. Let $v = s_{e+1}s_{e+2}\cdots s_f \in S$. The word $s_1 \cdots s_e v s_{f+1} \cdots s_k$ is a minimal weight representative for g (of length $e + 1 + k - f$). Since the function patt cannot increase weights, and all non-empty words have positive weight, we have

$$\omega(\text{patt}(s_1 \cdots s_e v s_{f+1} \cdots s_k)) \leq \omega(s_1 \cdots s_e v s_{f+1} \cdots s_k) = \omega(g) - \omega(v) \leq \omega(g) - 1.$$

Similarly, patt cannot increase lengths, so

$$\begin{aligned} |\text{patt}(s_1 \cdots s_e v s_{f+1} \cdots s_k)|_S &= |\text{patt}(s_1 \cdots s_e s_{f+1} \cdots s_k)|_S \\ &\leq |s_1 \cdots s_e s_{f+1} \cdots s_k|_S = e + k - f. \end{aligned}$$

We have $e - f < 0$, and by hypothesis $k \leq L$. So the height can be bounded above:

$$\mu_L(s_1 \cdots s_e v s_{f+1} \cdots s_k) \leq (\omega(g) - 1)L + e + k - f < \omega(g)L - L + k \leq \omega(g)L$$

and so we have again found a minimal weight word that represents g with strictly smaller height than st. $\square$

Proof of Proposition 3.1.5. Let $L = \max\{|st|_\Sigma \mid s, t \in Y\}$. Choose $\sigma = t_1 \cdots t_k \in S^*$ which is a minimal weight representative for g, and such that σ has minimal height μ_L amongst all minimal weight representatives for g (such a word always exists but may not be unique). We show that the pattern of this word has length at most d.

Suppose on the contrary that $l := |\text{patt}(\sigma)|_S > d$, and say the l elements of Y

appearing in σ are $t_{i_1}, t_{i_2}, \cdots, t_{i_l}$ (in that order). By Lemma (3.1.6), the assumption that σ has minimal height means that none of the t_{i_j} can be adjacent (otherwise we could find another minimal weight representative for g with smaller height).

Now consider the product

$$(t_{i_1} \cdots t_{i_2 - 1})(t_{i_2} \cdots t_{i_3 - 1}) \cdots (t_{i_d} \cdots t_{i_{d+1} - 1}).$$

By Lemma (3.1.2) some sub-product must be in $\mathbb{Z}^n$, say $(t_{i_e} \cdots t_{i_{e+1}-1}) \cdots (t_{i_f} \cdots t_{i_{f+1}-1})$, for $1 \leq e \leq f \leq d < l$. Rearrange this into a product of just two things: $(t_{i_e} \cdots t_{i_f})(t_{i_f+1} \cdots t_{i_{f+1}-1})$. The second factor is in $\mathbb{Z}^n$ since it lies between consecutive elements of $\mathrm{patt}(\sigma)$, and so the first factor must be in $\mathbb{Z}^n$ as well. Therefore these two factors commute, and we can switch them in σ without changing the weight or height. This leaves a word where t_{i_f} is adjacent to $t_{i_{f+1}}$. But this contradicts Lemma (3.1.6), so l is at most d. $\square$

This leads us to make the following definition.

Definition 3.1.7. For a virtually abelian group G, let P be the finite set of patterns of length at most d with respect to the extended generating set S:

$$P := \{\pi \in Y^* \mid |\pi|_S \leq d\}.$$

We have seen that, given any finite generating set Σ, we can pass to an extended generating set S while preserving the weighted length of all group elements. From now on, we will implicitly work with S, and make essential use of the fact (via Proposition 3.1.5) that any $g \in G$ can be represented by a minimal weight word with a pattern in the finite set P.

3.1.2 Structure constants for computing standard forms

Definition 3.1.8. For a pattern $\pi = y_{i_1} y_{i_2} \cdots y_{i_k}$ of length k, define the set of π-patterned words in S^* as

$$W^\pi = \{x_1^{w_1} \cdots x_r^{w_r} y_{i_1} x_1^{w_{r+1}} \cdots x_r^{w_{2r}} y_{i_2} x_1^{w_{2r+1}} \cdots y_{i_k} x_1^{w_{kr+1}} \cdots x_r^{w_{kr+r}} \mid w_j \in \mathbb{N}\}.$$

Any group element represented by a word with pattern π can be represented by a word in W^π (since powers of elements of X commute). For a fixed pattern π, W^π is in one-to-one correspondence with non-negative integer vectors of length $kr + r$. We define $m(\pi) = kr + r$. When it is clear which pattern this refers to, we will just write m.

Definition 3.1.9. Define a map $\varphi\colon W^\pi \to \mathbb{N}^m$ via

$$x_1^{w_1} \cdots x_r^{w_r} y_{i_1} x_1^{w_{r+1}} \cdots x_r^{w_{2r}} y_{i_2} x_1^{w_{2r+1}} \cdots y_{i_k} x_1^{w_{kr+1}} \cdots x_r^{w_{kr+r}} \mapsto \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix},$$

which records the powers of the generators contained in $\mathbb{Z}^n$. For $w \in W^\pi$, write $\vec{w} := \varphi(w)$, and for a subset $V \subseteq W^\pi$, write

$$\overrightarrow{V} := \varphi(V) \in \mathbb{N}^m.$$

Note that φ is a bijection.

Let $w \in W^\pi$ with the above form. Then the weight of w is given by

$$\omega(w) = \sum_{j=0}^{k} \sum_{i=1}^{r} \omega(x_i) w_{jr+i} + \sum_{j=1}^{k} \omega(y_{i_j}). \tag{3.2}$$

Let $T \subset G$ be a choice of representatives for the cosets $\mathbb{Z}^n \backslash G$. We can then express an element of G uniquely as $(a_1, \ldots, a_n)^\top t$ for $a_j \in \mathbb{Z}$, $t \in T$. In order to pass from a patterned word to this standard form, we introduce some constants.

Notation 3.1.10. For each $x_i \in X$, let $x_i = (z_{1i}, z_{2i}, \ldots, z_{ni})^\top \in \mathbb{Z}^n$, and define the $n \times r$ matrix

$$Z := [z_{ji}]_{1 \le j \le n, 1 \le i \le r} = \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{pmatrix},$$

which encodes the abelian part of the generating set.

We then have

$$x_1^{w_1} x_2^{w_2} \cdots x_r^{w_r} = Z \begin{pmatrix} w_1 \\ \vdots \\ w_r \end{pmatrix} \in \mathbb{Z}^n \tag{3.3}$$

for any powers $w_i \in \mathbb{N}$.

Now we encode conjugation of elements of $\mathbb{Z}^n$ via matrix multiplication. Let $e_i \in \mathbb{Z}^n$ be the ith standard basis vector, and $y_k \in Y$. Then $y_k e_i y_k^{-1} \in \mathbb{Z}^n \lhd G$, and we will let $y_k e_i y_k^{-1} = (\gamma_{1i,k}, \gamma_{2i,k}, \ldots, \gamma_{ni,k})$ for each $1 \leq i \leq n$, $y_k \in Y$. Define the $n \times n$ matrix

$$\Gamma_k := [\gamma_{ji,k}]_{1 \leq i,j \leq n} = \begin{pmatrix} \text{——} y_k e_1 y_k^{-1} \text{——} \\ \text{——} y_k e_2 y_k^{-1} \text{——} \\ \vdots \\ \text{——} y_k e_n y_k^{-1} \text{——} \end{pmatrix}^\top .$$

Then

$$y_k \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} y_k^{-1} = \Gamma_k \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \tag{3.4}$$

for any $y_k \in Y$, $a_i \in \mathbb{Z}$.

So we can express powers of the x_i generators as vectors in $\mathbb{Z}^n$, and we can move powers of y_k past such vectors by using the identity $y_k(a_1, \ldots, a_n) = \Gamma_k(a_1, \ldots, a_n)^\top y_k$. For the word

$$w = x_1^{w_1} x_2^{w_2} \cdots x_r^{w_r} y_{i_1} x_1^{w_{r+1}} \cdots x_r^{w_{2r}} y_{i_2} x_1^{w_{2r+1}} \cdots y_{i_k} x_1^{w_{kr+1}} \cdots x_r^{w_{kr+r}},$$

we can use the identities (3.3) and (3.4) to move all the y_k generators to the right, modifying the powers of x_i as we go, without changing the element that is represented. Thus

$$\overline{w} = \left\{ Z \begin{pmatrix} w_1 \\ \vdots \\ w_r \end{pmatrix} + \Gamma_{i_1} Z \begin{pmatrix} w_{r+1} \\ \vdots \\ w_{2r} \end{pmatrix} + \cdots + \Gamma_{i_1} \Gamma_{i_2} \cdots \Gamma_{i_k} Z \begin{pmatrix} w_{kr+1} \\ \vdots \\ w_m \end{pmatrix} \right\} \overline{\pi} \tag{3.5}$$

To express this more compactly, we introduce further notation.

Definition 3.1.11. Consider the $n \times m$ matrix formed by placing the matrices Z, $\Gamma_{i_1} Z$, $\Gamma_{i_1}\Gamma_{i_2} Z \dots$ next to each other. The transposes of the rows of this new matrix are m-dimensional vectors which we will call A_i^π as follows:

$$\begin{pmatrix} \text{---} (A_1^\pi)^\top \text{---} \\ \text{---} (A_2^\pi)^\top \text{---} \\ \vdots \\ \text{---} (A_n^\pi)^\top \text{---} \end{pmatrix} := \left(\begin{array}{c|c|c|c|c} Z & \Gamma_{i_1} Z & \Gamma_{i_1}\Gamma_{i_2} Z & \cdots & \Gamma_{i_1}\Gamma_{i_2}\cdots\Gamma_{i_k} Z \end{array} \right).$$

Definition 3.1.12. The word π itself represents some element of G, so we introduce integers B_i^π so that we can write $\overline{\pi}$ in the standard form given by the choice of coset representatives as

$$\overline{\pi} = \begin{pmatrix} B_1^\pi \\ B_2^\pi \\ \vdots \\ B_n^\pi \end{pmatrix} t_\pi,$$

for $t_\pi \in T$.

Now we may rewrite equation (3.5) using scalar products as

$$\overline{w} = \left\{ \begin{pmatrix} A_1^\pi \cdot \vec{w} \\ A_2^\pi \cdot \vec{w} \\ \vdots \\ A_n^\pi \cdot \vec{w} \end{pmatrix} + \begin{pmatrix} B_1^\pi \\ B_2^\pi \\ \vdots \\ B_n^\pi \end{pmatrix} \right\} t_\pi. \tag{3.6}$$

So words in W^π represent the same element of G if and only if the scalar products of the corresponding vectors with the A_i^πs agree.

Remark 3.1.13. *We emphasise that $A_i^\pi \in \mathbb{Z}^m$, $B_i^\pi \in \mathbb{Z}$, and $t_\pi \in G$ are constant in the sense that they depend only on the pattern π. In particular, any two words with the same pattern represent elements of the same coset.*

Definition 3.1.14. For a pattern $\pi = y_{i_1} \cdots y_{i_k}$ of length k, let $A_{n+1}^\pi \in \mathbb{N}^m$ record the weights of the x_i generators, ordered as follows:

$$A^{\pi}_{n+1} = \underbrace{(\omega(x_1), \omega(x_2), \ldots, \omega(x_r), \ldots, \omega(x_1), \omega(x_2), \ldots, \omega(x_r))^{\top}}_{r \text{ weights, repeated } k+1 \text{ times, giving an } m\text{-dimensional vector}}.$$

Furthermore, let B^{π}_{n+1} record the weight of the word π, i.e.

$$B^{\pi}_{n+1} = \sum_{j=1}^{k} \omega(y_{i_j})$$

We can then express equation (3.2) more compactly using a scalar product:

$$\omega(w) = A^{\pi}_{n+1} \cdot \vec{w} + B^{\pi}_{n+1}. \tag{3.7}$$

We now have a collection of vectors A^{π}_i and integers B^{π}_i which together put words in W^{π} into the chosen standard form.

3.1.3 Structure constants for testing conjugacy

In what follows we introduce notation that we will need to prove Theorem 3.3.1. As above, we fix a transversal T for $\mathbb{Z}^n \backslash G$.

In a similar manner to above, we encode conjugation of an element

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{Z}^n \lhd G$$

by some other element of G via multiplication by a matrix. Let $t \in T$, and e_i be the ith standard basis vector in $\mathbb{Z}^n$. Then $te_it^{-1} \in \mathbb{Z}^n$ by normality. Let $te_it^{-1} = (\delta_{1i,t}, \delta_{2i,t}, \ldots, \delta_{ni,t})^{\top}$, and write Δ_t for the $n \times n$ matrix $[\delta_{ji,t}]_{1 \le i,j \le n}$ whose columns are te_it^{-1}. Then we have

$$t \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} t^{-1} = \Delta_t \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

for any $a_i \in \mathbb{Z}$.

Fix a pattern π. Recall the matrices Z, and Γ_j for each element $y_j \in Y$. If

$$w = x_1^{w_1} x_2^{w_2} \cdots x_r^{w_r} y_{j_1} x_1^{w_{r+1}} x_2^{w_{r+2}} \cdots x_r^{w_{2r}} y_{i_2} \cdots y_{i_k} x_1^{w_{kr+1}} \cdots x_r^{w_m} \in W^\pi,$$

consider the group element $t\overline{w}t^{-1}$ for $t \in T$. With Δ_t as defined above, equation (3.5) yields

$$t\overline{w}t^{-1} = \left\{ \Delta_t Z \begin{pmatrix} w_1 \\ \vdots \\ w_r \end{pmatrix} + \Delta_t \Gamma_{j_1} Z \begin{pmatrix} w_{r+1} \\ \vdots \\ w_{2r} \end{pmatrix} + \cdots \right.$$
$$\left. + \Delta_t \Gamma_{j_1} \cdots \Gamma_{j_k} Z \begin{pmatrix} w_{kr+1} \\ \vdots \\ w_m \end{pmatrix} \right\} t\overline{\pi}t^{-1}. \tag{3.8}$$

We make definitions analogous to 3.1.11 and 3.1.12 above.

Definition 3.1.15. For each pattern π, and $t \in T$, place the matrix products in order and define m-dimensional vectors $A^\pi_{i,t}$ as follows

$$\begin{pmatrix} \text{——} \left(A^\pi_{1,t}\right)^\top \text{——} \\ \text{——} \left(A^\pi_{2,t}\right)^\top \text{——} \\ \vdots \\ \text{——} \left(A^\pi_{n,t}\right)^\top \text{——} \end{pmatrix} := \left(\Delta_t Z \,\middle|\, \Delta_t \Gamma_{j_1} Z \,\middle|\, \cdots \,\middle|\, \Delta_t \Gamma_{j_1} \ldots \Gamma_{j_k} Z \right).$$

Now we can express equation (3.8) in terms of scalar products:

$$t\overline{w}t^{-1} = \begin{pmatrix} A^\pi_{1,t} \cdot \vec{w} \\ A^\pi_{2,t} \cdot \vec{w} \\ \vdots \\ A^\pi_{n,t} \cdot \vec{w} \end{pmatrix} t\overline{\pi}t^{-1}.$$

Definition 3.1.16. We have $t\overline{\pi}t^{-1} = xs$ for some $s \in T$, $x \in \mathbb{Z}^n$. This x depends only on π and t. Write $x = \left(B^\pi_{1,t}, \ldots, B^\pi_{n,t}\right)^\top$.

Thus we have

$$t\overline{w}t^{-1} = \left\{ \begin{pmatrix} A^{\pi}_{1,t} \cdot \vec{w} \\ A^{\pi}_{2,t} \cdot \vec{w} \\ \vdots \\ A^{\pi}_{n,t} \cdot \vec{w} \end{pmatrix} + \begin{pmatrix} B^{\pi}_{1,t} \\ B^{\pi}_{2,t} \\ \vdots \\ B^{\pi}_{n,t} \end{pmatrix} \right\} s. \tag{3.9}$$

Now we have a collection of vectors that we can use to test whether two words represent conjugate elements (provided we know that their coset representatives are conjugate). The following Lemma makes this precise.

Lemma 3.1.17. *Let $v \in W^{\pi}$ and $w \in W^{\mu}$ for patterns π and μ, and let $t \in T$. Then $\overline{v} = t\overline{w}t^{-1}$ if and only if*

1. *$\overline{\pi}$ and $t\overline{\mu}t^{-1}$ are in the same $\mathbb{Z}^n$-coset, and*

2. *$A^{\pi}_{i} \cdot \vec{v} + B^{\pi}_{i} = A^{\mu}_{i,t} \cdot \vec{w} + B^{\mu}_{i,t}$ for each $1 \leq i \leq n$.*

Proof. From equation (3.6) we have

$$\overline{v} = \left\{ (A^{\pi}_{1} \cdot \vec{w}, A^{\pi}_{2} \cdot \vec{w}, \ldots, A^{\pi}_{n} \cdot \vec{w})^{\top} + (B^{\pi}_{1}, B^{\pi}_{2}, \ldots, B^{\pi}_{n})^{\top} \right\} t_{\pi}.$$

From equation (3.9) we have

$$t\overline{w}t^{-1} = \left\{ \left(A^{\mu}_{1,t} \cdot \vec{w}, A^{\mu}_{2,t} \cdot \vec{w}, \ldots, A^{\mu}_{n,t} \cdot \vec{w}\right)^{\top} + \left(B^{\mu}_{1,t}, B^{\mu}_{2,t}, \ldots, B^{\mu}_{n,t}\right)^{\top} \right\} s$$

where s is determined by the element $t\overline{\mu}t^{-1}$, and the result follows. □

3.1.4 Polyhedral Sets

The following definition and results follow Benson's work [4]. However, the ideas appear in model theory as early as Presburger [52]. Results regarding rationality can be found in [21], and the ideas appear in the theory of Igusa local zeta functions (see [16]). These last are linked to subgroup growth [39], a different notion of growth in groups not considered in this thesis.

Definition 3.1.18. Let $m \in \mathbb{N}_{+}$, and let $\cdot$ denote the Euclidean scalar product. Then for any $\boldsymbol{u} \in \mathbb{Z}^m$, $a \in \mathbb{Z}$, $b \in \mathbb{N}_{+}$:

1. an *elementary set* is any subset of $\mathbb{Z}^m$ of the form $\{\boldsymbol{z} \in \mathbb{Z}^m \mid \boldsymbol{u} \cdot \boldsymbol{z} = a\}$, $\{\boldsymbol{z} \in \mathbb{Z}^m \mid \boldsymbol{u} \cdot \boldsymbol{z} > a\}$, or $\{\boldsymbol{z} \in \mathbb{Z}^m \mid \boldsymbol{u} \cdot \boldsymbol{z} \equiv a \mod b\}$,

2. a *basic polyhedral set* is any finite intersection of elementary sets,

3. a *polyhedral set* is any finite union of basic polyhedral sets.

If $\mathcal{P} \subset \mathbb{Z}^m$ is polyhedral and additionally $\mathcal{P} \subseteq \mathbb{N}^m$, we call $\mathcal{P}$ a *positive polyhedral set.*

We record some crucial facts about polyhedral sets.

Proposition 3.1.19 (Proposition 13.1 of [4])**.** *Polyhedral sets in $\mathbb{Z}^m$ are closed under finite unions, finite intersections, and set complement.*

Proposition 3.1.20 (Propositions 13.7 and 13.8 of [4])**.** *Let $\mathcal{E} \colon \mathbb{Z}^m \to \mathbb{Z}^{m'}$ be an integral affine transformation (for some $m, m' > 0$). That is, there is some $m' \times m$ matrix with integer entries and some $q \in \mathbb{Z}^{m'}$ such that $\mathcal{E}(p) = Ap + q$ for $p \in \mathbb{Z}^m$. If $\mathcal{P} \subseteq \mathbb{Z}^m$ is a polyhedral set then $\mathcal{E}(\mathcal{P}) \subseteq \mathbb{Z}^{m'}$ is a polyhedral set. If $\mathcal{Q} \subseteq \mathbb{Z}^{m'}$ is a polyhedral set then the preimage $\mathcal{E}^{-1}(\mathcal{Q}) \subseteq \mathbb{Z}^m$ is a polyhedral set.*

We note that projection onto any subset of the coordinates of $\mathbb{Z}^m$ is an integral affine transformation.

Let $\mathcal{P} \subseteq \mathbb{N}^m$ be a positive polyhedral set. Given some choice of weights $(\omega_1, \ldots, \omega_m)$ for the coordinates of $\mathbb{N}^m$, we assign the weight $\sum_{i=1}^m a_i\omega_i$ to the vector $(a_1, \ldots, a_m)^\top \in \mathcal{P}$. Define

$$\sigma^\omega_{\mathcal{P}}(n) = \#\{p \in \mathcal{P} \mid \omega(p) = n\},$$

and the resulting weighted growth series

$$\mathcal{S}^\omega_{\mathcal{P}}(z) = \sum_{n=0}^{\infty} \sigma^\omega_{\mathcal{P}}(n) z^n.$$

Proposition 3.1.21 (Proposition 14.1 of [4], and Lemma 7.5 of [21])**.** *If $\mathcal{P}$ is a positive polyhedral set, the weighted growth series $\mathcal{S}^\omega_{\mathcal{P}}(z)$ is a rational function.*

We will need the following Lemma concerning polyhedral sets.

Lemma 3.1.22. *Let $\mathcal{P}$ be a polyhedral subset of $\mathbb{Z}^m$ for some $m \geq 1$. Suppose there exist polyhedral sets $X_1, \ldots, X_k \subset \mathbb{Z}^m$ such that $\mathcal{P} \subseteq \bigcup_{i=1}^k X_i$. Then there exist polyhedral sets $Y_i \subseteq X_i$ for each i such that $Y_i \cap Y_j = \emptyset$ for $i \neq j$ and $\mathcal{P} = \bigcup_{i=1}^k Y_i$.*

Proof. We induct on k. Consider $k = 1$, i.e. $\mathcal{P} \subseteq X_1$. Let $Y_1 = \mathcal{P} \cap X_1$, which is polyhedral as an intersection of polyhedral sets. In other words $\mathcal{P} = Y_1$.

Now assume the statement is true for some $k > 1$. Let $\mathcal{P}$ be some polyhedral set, with polyhedral sets $X_1, \ldots, X_{k+1}$ such that $\mathcal{P} \subseteq \bigcup_{i=1}^{k+1} X_i$. Consider the polyhedral set

$$\mathcal{Q} := \mathcal{P} \cap \bigcup_{i=1}^{k} X_i.$$

Now since $\mathcal{Q} \subseteq \bigcup_{i=1}^{k} X_i$, the inductive hypothesis gives polyhedral sets $Y_i \subseteq X_i$ for $1 \leq i \leq k$ such that $Y_i \cap Y_j = \emptyset$ for each $i \neq j$, $1 \leq i, j \leq k$, and

$$\mathcal{Q} = \bigcup_{i=1}^{k} Y_i.$$

Now let $Y_{k+1} = \mathcal{P} \setminus \mathcal{Q}$, also a polyhedral set. We have $Y_{k+1} \cap \mathcal{Q} = \emptyset$, and $Y_{k+1} \subseteq X_{k+1}$ (since the X_is cover $\mathcal{P}$ and Y_{k+1} does not intersect $\mathcal{Q}$), and by definition

$$\mathcal{P} = \mathcal{Q} \cup Y_{k+1} = \bigcup_{i=1}^{k+1} Y_i.$$

So the statement holds for $k + 1$. $\square$

3.1.5 N-fold patterns

We develop a framework for dealing with N-tuples of patterned words for some $N \in \mathbb{N}_+$.

Definition 3.1.23. Let Q be any set of patterns (for a virtually abelian group with some choice of finite generating set). Let N be a positive integer. An *N-fold pattern* will be an N-tuple of patterns from Q. We will write $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_N) \in Q^N$.

Given an N-fold pattern $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)$, write $W^{\boldsymbol{\pi}}$ for the set of N-tuples of words with patterns given by $\boldsymbol{\pi}$. More precisely,

$$W^{\boldsymbol{\pi}} = \left\{ \left(w^{(1)}, w^{(2)}, \ldots, w^{(N)}\right) \,\middle|\, w^{(i)} \in W^{\pi_i},\ 1 \leq i \leq N \right\}.$$

Let $m(\boldsymbol{\pi}) = \sum_i m(\pi_i)$. As above, we extract the powers of the x_i generators with respect to each pattern and note that elements of $W^{\boldsymbol{\pi}}$ are in one-to-one correspondence with vectors in $\mathbb{N}^{m(\boldsymbol{\pi})}$ via

$$\left(w^{(1)}, \ldots, w^{(N)}\right) \mapsto \begin{pmatrix} \overrightarrow{w^{(1)}} \\ \vdots \\ \overrightarrow{w^{(N)}} \end{pmatrix}.$$

In the following Lemma, we show that if we have a polyhedral set of N-tuples in $\mathbb{N}^{m(\boldsymbol{\pi})}$, we can extract a minimal-weight element of each tuple, in a manner that preserves rational growth.

Lemma 3.1.24. *Let G be virtually abelian, generated by S, with weight function ω. Fix some N-fold pattern $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)$. Let $V \subset W^{\boldsymbol{\pi}}$. If $\overrightarrow{V}$ is a polyhedral set then there exists a language $\mathcal{L} \subset S^*$ in one-to-one correspondence with V such that $\mathcal{L}$ consists of exactly one element $w \in \{v_1, \ldots, v_N\}$, for each $(v_1, \ldots, v_N) \in V$, with $\omega(w) \leq \omega(v_i)$ for each $1 \leq i \leq N$, and such that $\mathcal{L}$ has rational weighted growth series.*

Proof. Let

$$X_j(\boldsymbol{\pi}) = \left\{ \left(w^{(1)}, \ldots, w^{(N)}\right) \in W^{\boldsymbol{\pi}} \,\middle|\, \omega\left(w^{(j)}\right) \leq \omega\left(w^{(k)}\right),\ 1 \leq k \leq N \right\}$$

be those elements in $W^{\boldsymbol{\pi}}$ where the jth component word has minimal weight. For each π_k, define the $m(\boldsymbol{\pi})$-dimensional vector D^{π_k} as follows:

$$D^{\pi_k} := \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ A^{\pi_k}_{n+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \sum_{i=1}^{k-1} m(\pi_i) \text{ zeros} \\ m(\pi_k) \text{ rows} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \sum_{i=k+1}^{N} m(\pi_i) \text{ zeros} \end{array}$$

Then for some N-tuple of words $\left(w^{(1)}, \ldots, w^{(N)}\right) \in W^{\boldsymbol{\pi}}$, whose corresponding vector is $\vec{z} \in \mathbb{N}^{m(\boldsymbol{\pi})}$, we have $D^{\pi_k} \cdot \vec{z} = A^{\pi_k}_{n+1} \cdot \overrightarrow{w^{(k)}}$.

For $w \in W^{\pi_j}$ and $w' \in W^{\pi_k}$, equation (3.7) implies that

$$\omega(w) \leq \omega(w') \Leftrightarrow A_{n+1}^{\pi_j} \cdot \vec{w} + B_{n+1}^{\pi_j} \leq A_{n+1}^{\pi_k} \cdot \vec{w'} + B_{n+1}^{\pi_k}.$$

Therefore we have

$$\begin{aligned} \overrightarrow{X_j(\boldsymbol{\pi})} &= \bigcap_{\substack{1 \leq k \leq N \\ k \neq j}} \left\{ \vec{z} \in \mathbb{N}^{m(\boldsymbol{\pi})} \mid D^{\pi_j} \cdot \vec{z} + B_{n+1}^{\pi_j} \leq D^{\pi_k} \cdot \vec{z} + B_{n+1}^{\pi_k} \right\} \\ &= \bigcap_{\substack{1 \leq k \leq N \\ k \neq j}} \left\{ \vec{z} \in \mathbb{N}^{m(\boldsymbol{\pi})} \mid (D^{\pi_j} - D^{\pi_k}) \cdot \vec{z} \leq B_{n+1}^{\pi_k} - B_{n+1}^{\pi_j} \right\}. \end{aligned}$$

Thus each $\overrightarrow{X_j(\boldsymbol{\pi})}$ is a polyhedral subset of $\mathbb{N}^{m(\boldsymbol{\pi})}$ (since it is a finite intersection of elementary sets). Since every N-tuple in V must have at least one minimal-weight element, we have $\vec{V} \subseteq \bigcup_{j=1}^{N} \overrightarrow{X_j(\boldsymbol{\pi})}$. Lemma 3.1.22 then gives us polyhedral sets $\overrightarrow{Y_j(\boldsymbol{\pi})} \subseteq \overrightarrow{X_j(\boldsymbol{\pi})}$ for each j which are pairwise disjoint, and such that $\vec{V} = \bigcup_{j=1}^{N} \overrightarrow{Y_j(\boldsymbol{\pi})}$.

Let $\rho_j \colon \mathbb{N}^{m(\boldsymbol{\pi})} \to \mathbb{N}^{m(\pi_j)}$ be the projection onto just the coordinates corresponding to the jth component of the N-tuple. Since images of polyhedral sets under projections are still polyhedral by Proposition 3.1.20, we have N polyhedral sets of the form $\rho_j(\overrightarrow{Y_j(\boldsymbol{\pi})})$, each corresponding to a collection of words which are minimal-weight representatives for their N-tuple, and each growing rationally with respect to the generators of G (by Proposition 3.1.21). Since the $\vec{Y_j}$s are disjoint, and cover $\vec{V}$, the union of these polyhedral sets corresponds to a language of unique, minimal-weight representatives for V, which grows rationally. □

3.1.6 Dickson's Lemma

In what follows we will need a version of Dickson's Lemma.

Definition 3.1.25. Let $H = \{H_1, \ldots, H_m\}$ be a basis for $\mathbb{Z}^m$. Define an ordering on $\mathbb{Z}^m$ by comparing the components pairwise. That is, $v >_H w$ iff $H_i \cdot v \geq H_i \cdot w$ for all i.

Note that this is a partial order, but not a total order. Consider the following class of bases. Let $I \subseteq \{1, 2, \ldots, m\}$ be any subset. Then define $H_i^I = \begin{cases} e_i & i \in I \\ -e_i & i \notin I \end{cases}$.

Let $H^I = \{H_1^I, \ldots, H_m^I\}$, which is a basis for $\mathbb{Z}^m$. The following is a version of Dickson's Lemma (see [22]):

Lemma 3.1.26. *Given some subset $I \subseteq \{1, \ldots, m\}$ as above, let $Q_I^m = \{v \in \mathbb{Z}^m \mid v >_{H^I} (0, \ldots, 0)\}$ be the 'non-negative' quadrant with respect to the H^I-ordering. Then any subset, V^m, of Q_I^m contains a finite subset U with the property that for each $v \in V^m$ there exists some $u \in U$ with $v >_H u$. In other words, every subset of Q_I^m has only a finite number of minimal elements.*

To prove this, we need the following Lemma:

Lemma 3.1.27. *Given $I \subseteq \{1, \ldots, m\}$, any infinite sequence of vectors in $Q_I^m \subset \mathbb{Z}^m$ has in infinite, non-decreasing subsequence, with respect to H^I.*

Proof. We proceed by induction on m. For $m = 1$, H^I is either $\{1\}$ or $\{-1\}$. In the positive case Q_I^1 consists of the non-negative integers, and the ordering is the usual one. Given a sequence $\{a_i\}_{i \in \mathbb{N}} \subset V$, obtain a non-decreasing subsequence by omitting those a_i for which there is some $j < i$ with $a_j > a_i$. This is infinite since the original sequence can only decrease to zero, and at worst we have an infinite sequence of zeros. In the negative case, $a >_H a'$ if and only if $a \leq a'$, and Q_I^1 is the non-positive integers. The same argument with flipped directions gives the required subsequence.

Now assume that the statement is true for $\mathbb{Z}^m$. Consider Q_I^{m+1} with respect to a basis H^I. Let

$$\{\mathbf{a}^i\}_{i \in \mathbb{N}} = \{(a_1^i, a_2^i, \ldots, a_{m+1}^i)\}_{i \in \mathbb{N}}$$

be an infinite sequence in Q_I^{m+1}. The projection of each term of the sequence onto the first m coordinates forms an infinite sequence in Q_J^m, where $J = I \setminus \{m+1\}$ (which may equal I). By the inductive hypothesis this sequence has an infinite subsequence of non-decreasing terms with respect to H^J, say

$$\{\mathbf{a}^{i_j}\}_{j \in \mathbb{N}} = \{(a_1^{i_j}, a_2^{i_j}, \ldots, a_m^{i_j})\}_{j \in \mathbb{N}}.$$

The corresponding infinite sequence $\{a_{m+1}^{i_j}\}_{j \in \mathbb{N}}$ then has an infinite subsequence of terms that are non-decreasing with respect to the $m+1$th basis vector in H^I (which is 1 or -1 according to whether or not $m+1 \in I$). Call this subsequence

$\{a_{m+1}^{i_{j_k}}\}_{k\in\mathbb{N}}$. We then take the infinite sequence $\{(a_1^{i_{j_k}}, \dots, a_{m+1}^{i_{j_k}}\}_{k\in\mathbb{N}} \subset V^{m+1}$, which is non-decreasing with respect to H^I, proving the lemma. $\square$

Proof of Lemma 3.1.26. Suppose that the set of minimal elements U is not finite. No pair of these elements is comparable, since otherwise we could omit the one which was greater and still have a set of minimal elements. Since $U \subset \mathbb{Z}^m$ is countable, we can think of it as an infinite sequence. But then Lemma 3.1.27 gives an infinite subsequence of elements that are non-decreasing, so in particular can be compared pairwise. This is a contradiction, therefore U is finite. $\square$

3.2 Coset Growth series

In this section we demonstrate the following:

Theorem 3.2.1. *Let G be a virtually abelian group with any choice of finite, weighted generating set S, and H any subgroup of G. Then the set of right cosets $H\backslash G$ has rational weighted growth series with respect to S.*

This is proved in two parts. Firstly, we generalise the main result of [4] to show that if a subgroup F is contained in a finite index abelian subgroup of G then there exists a language of minimal representatives for the cosets of F in G that grows rationally.

Secondly, for a general subgroup H, its intersection $F(H)$ with the finite index abelian subgroup of G is abelian, and of finite index in H. So we consider the language of minimal representatives for the cosets of $F(H)$ in G, and from these we choose a language of representatives for the cosets of H. We show that this language can be chosen so that it grows rationally.

The first step is the following special case of Theorem 3.2.1.

Theorem 3.2.2. *Let G be a virtually abelian group, with any choice of finite weighted generating set S, and with finite index normal subgroup $\mathbb{Z}^n$. If F is a subgroup of G contained in $\mathbb{Z}^n$, then the weighted growth of $F\backslash G$ with respect to S is rational.*

Theorem 1.2 of [4] is the case where F is the trivial group. The proof given here follows a similar structure to that in [4]. For each pattern $\pi \in P$, we establish an ordering on the words of W^π which respects their weight, and use this to find a polyhedral set of minimal representatives for the cosets that intersect $\overline{W^\pi}$. We then show that the overlaps between these sets are also polyhedral, so can be removed while preserving rationality. Note that if F has finite index in G, $F\backslash G$ is finite and so the growth series is a polynomial, and so trivially rational. From now on we assume that F has infinite index in G, and hence $[\mathbb{Z}^n \colon F]$ is also infinite.

We first establish a criterion for when two words represent elements of the same F-coset. We will again use the vectors $A_1^\pi, \ldots, A_n^\pi$ (of dimension $m(\pi)$), $A_1^\mu, \ldots, A_n^\mu$ (of dimension $m(\mu)$) and integers $B_1^\pi, \ldots, B_n^\pi$, and $B_1^\mu \ldots, B_n^\mu$, of Section 3.1.

Proposition 3.2.3. *Let $v \in W^\pi$, $w \in W^\mu$, for some patterns π,μ. Let $F < \mathbb{Z}^n$ be of rank $f \leq n$, with basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_f\} \subset \mathbb{Z}^n$. Then $\overline{v}$ and $\overline{w}$ are in the same coset of F in G if and only if*

1. *$\overline{\pi}$ and $\overline{\mu}$ are in the same coset of $\mathbb{Z}^n$ in G, and*

2. *there exist integers $a_1, \ldots, a_f$ such that*

$$A_i^\pi \cdot \vec{v} + B_i^\pi - (A_i^\mu \cdot \vec{w} + B_i^\mu) = e_i \cdot \left(\sum_{j=1}^{f} a_j \boldsymbol{b}_j \right)$$

   for each $1 \leq i \leq n$ (where e_i as usual denotes the ith standard basis vector in $\mathbb{Z}^n$).

Proof. Recall (see (3.6)) that the group elements represented by v and w are given by

$$\overline{v} = \left\{ (A_1^\pi \cdot \vec{v}, A_2^\pi \cdot \vec{v}, \ldots, A_n^\pi \cdot \vec{v})^\top + (B_1^\pi, B_2^\pi, \ldots, B_n^\pi)^\top \right\} t_\pi \in G$$

and

$$\overline{w} = \left\{ (A_1^\mu \cdot \vec{w}, A_2^\mu \cdot \vec{w}, \ldots, A_n^\mu \cdot \vec{w})^\top + (B_1^\mu, B_2^\mu, \ldots, B_n^\mu)^\top \right\} t_\mu \in G$$

respectively, where t_π and t_μ are the chosen representatives for the cosets $\mathbb{Z}^n\overline{\pi}$ and $\mathbb{Z}^n\overline{\mu}$.

Now two words v, w represent elements of the same F-coset if and only if $\overline{v}(\overline{w})^{-1} \in F$. This is equivalent to the existence of integer coefficients $a_1, \ldots, a_f$ such that

$$\overline{v}(\overline{w})^{-1} = \sum_{j=1}^{f} a_j \boldsymbol{b}_j.$$

Suppose that our words v and w do represent the same F-coset. Since $Fg \subset \mathbb{Z}^n g$ for any $g \in G$, v and w represent the same $\mathbb{Z}^n$-coset, so $t_\pi = t_\mu$, which is precisely condition (1).

We have

$$\begin{aligned}\overline{v}(\overline{w})^{-1} = &\left\{(A_1^\pi \cdot \vec{v}, A_2^\pi \cdot \vec{v}, \ldots, a_n^\pi \cdot \vec{v})^\top + (B_1^\pi, B_2^\pi, \ldots, B_n^\pi)^\top\right\} t_\pi \\ &\cdot t_\mu^{-1} \left\{-(A_1^\mu \cdot \vec{w}, A_2^\mu \cdot \vec{w}, \ldots, A_n^\mu \cdot \vec{w})^\top - (B_1^\mu, B_2^\mu, \ldots, B_n^\mu)^\top\right\} \\ = &(A_1^\pi \cdot \vec{v} + B_1^\pi - A_1^\mu \cdot \vec{w} - B_1^\mu, \ldots, A_n^\pi \cdot \vec{v} + B_n^\pi - A_n^\mu \cdot \vec{w} - B_n^\mu)^\top.\end{aligned}$$

Thus

$$(A_1^\pi \cdot \vec{v} + B_1^\pi - A_1^\mu \cdot \vec{w} - B_1^\mu, \ldots, A_n^\pi \cdot \vec{v} + B_n^\pi - A_n^\mu \cdot \vec{w} - B_n^\mu)^\top = \sum_{j=1}^{f} a_j \boldsymbol{b}_j \in \mathbb{Z}^n,$$

which is equivalent to condition (2).

Conversely, if v and w satisfy the conditions, then $t_\pi = t_\mu$ and so

$$\overline{v}(\overline{w})^{-1} = (A_1^\pi \cdot \vec{v} + B_1^\pi - A_1^\mu \cdot \vec{w} - B_1^\mu, \ldots, A_n^\pi \cdot \vec{v} + B_n^\pi - A_n^\mu \cdot \vec{w} - B_n^\mu)^\top = \sum_{j=1}^{f} a_j \boldsymbol{b}_j$$

i.e. $\overline{v}(\overline{w})^{-1} \in F$. $\square$

We note the following special case, when $\pi = \mu$.

Corollary 3.2.4. *Let $v, w \in W^\pi$, $F < \mathbb{Z}^n$ with rank f and basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_f\} \subset \mathbb{Z}^n$. Then $\overline{v}$ and $\overline{w}$ are in the same F-coset if and only if there exist integers $a_1, \ldots a_f$ so that*

$$A_i^\pi \cdot (\vec{v} - \vec{w}) = e_i \cdot \sum_{j=1}^{f} a_j \boldsymbol{b}_j$$

for each $1 \leq i \leq n$.

We now build a polyhedral set of minimal weight coset representatives, for each pattern π. The following is a modification of the arguments in Section 6 of [4].

Proposition 3.2.5. *Fix a pattern $\pi \in P$, and with F an infinite index subgroup of $\mathbb{Z}^n \triangleleft G$, consider the set $(F \backslash G)^\pi$ of right cosets which contain an element represented* *by a word in W^π, i.e.*

$$(F\backslash G)^\pi = \{Fg \in F\backslash G \mid Fg \cap \overline{W^\pi} \neq \emptyset\}.$$

Then there exists a set $V_F^\pi \subset W^\pi$ consisting of minimal-weight (amongst W^π) representatives for every coset in $(F\backslash G)^\pi$, with the property that $\overrightarrow{V_F^\pi}$ is a polyhedral set.

Proof. We will define an ordering on words in W^π that is consistent with the weight ordering, and yields a unique minimal representative for each coset in $(F\backslash G)^\pi$.

Fix a basis $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_f\}$ for F. So that this proof applies in full generality, in the case that F is the trivial group we set $\boldsymbol{b}_1$ to be the zero vector and consider the 'basis' $\{\boldsymbol{b}_1\}$. Recall the vectors $A_1^\pi, \ldots, A_{n+1}^\pi \in \mathbb{Z}^m$. Choose standard basis vectors $A_{n+2}^\pi, \ldots, A_K^\pi$ so that $\{A_{n+1}^\pi, \ldots, A_K^\pi\}$ consists of m linearly independent vectors in $\mathbb{Z}^m$ (and so $K = n + m$). Note that $m \geq 1$ since $m = (k+1)|X|$ and $X = S \cap \mathbb{Z}^n$ is non-empty (see Definition 3.1.3). Define an order on the words of W^π as follows. We will write $v \leq_\pi w$ if and only if either $v = w$ or there exist integers $a_1, \ldots, a_f$, and $0 \leq i \leq m$ such that

$$A_k^\pi \cdot (\vec{v} - \vec{w}) = e_k \cdot \sum_{j=1}^{f} a_j \boldsymbol{b}_j, \text{ for } 1 \leq k \leq n,$$

$$A_k^\pi \cdot (\vec{v} - \vec{w}) = 0, \text{ for } n+1 \leq k < n+i, \text{ and}$$

$$A_{n+i}^\pi \cdot (\vec{v} - \vec{w}) > 0.$$

We show that this is a partial order on W^π. Firstly, note that the ordering is reflexive by definition. Next, we show transitivity. Suppose $u, v, w \in W^\pi$ with $u \leq_\pi v \leq_\pi w$. If $u = v$ or $v = w$ then clearly $u \leq_\pi w$, so suppose $u \neq v \neq w$. So we have integers

$a_1, \dots, a_f, a'_1 \dots, a'_f$ and $0 \le i, i' \le m$ with

$$A^\pi_k \cdot (\vec{u} - \vec{v}) = e_k \cdot \sum_{j=1}^{f} a_j \boldsymbol{b}_j, \text{ for } 1 \le k \le n,$$

$$A^\pi_k \cdot (\vec{u} - \vec{v}) = 0, \text{ for } n+1 \le k < n+i, \text{ and}$$

$$A^\pi_{n+i} \cdot (\vec{u} - \vec{v}) > 0$$

and

$$A^\pi_k \cdot (\vec{v} - \vec{w}) = e_k \cdot \sum_{j=1}^{f} a'_j \boldsymbol{b}_j, \text{ for } 1 \le k \le n,$$

$$A^\pi_k \cdot (\vec{v} - \vec{w}) = 0, \text{ for } n+1 \le k < n+i', \text{ and}$$

$$A^\pi_{n+i'} \cdot (\vec{v} - \vec{w}) > 0.$$

Then for $0 \le k \le n$ we have

$$A^\pi_k \cdot (\vec{u} - \vec{w}) = A^\pi_k \cdot (\vec{u} - \vec{v}) + A^\pi_k \cdot (\vec{v} - \vec{w}) = e_k \cdot \sum_{j=1}^{f} (a_j + a'_j) \boldsymbol{b}_j$$

by linearity of the scalar product. Furthermore, $A^\pi_k \cdot (\vec{u} - \vec{w}) = 0$ for $n+1 \le k < \min(i, i')$ and $A^\pi_k \cdot (\vec{u} - \vec{w}) > 0$ for $k = \min(i, i')$, and thus $u \le_\pi w$, so $\le_\pi$ is transitive. For antisymmetry, note that if $v \le_\pi w$ and $w \le_\pi v$ then either $v = w$ or we must have $A^\pi_k \cdot (\vec{v} - \vec{w}) = A^\pi_k \cdot (\vec{w} - \vec{v}) = 0$ for each $n+1 \le k \le K$, and since the corresponding A^π_ks consist of m linearly independent elements of $\mathbb{Z}^m$, we must have $\vec{v} = \vec{w}$, i.e. $v = w$. Thus the order is a well-defined partial order (although not a total order).

If we restrict ourselves to the words representing a single F-coset (understanding 'F-coset' to mean 'element' when F is trivial), this becomes a well-ordering. To see this, suppose we have an infinite descending chain of words in W^π that represent the same coset:

$$w_1 \ge_\pi w_2 \ge_\pi \cdots .$$

Since $A_k^\pi \in \mathbb{N}^m$ for $k > n$ and $\vec{w}_i \in \mathbb{N}^m$ for all i, the sequences

$$A_k^\pi \cdot \vec{w}_1 \geq A_k^\pi \cdot \vec{w}_2 \geq \cdots$$

for each $k > n$ consist of non-negative integers and so must stabilize, say after i_k steps. Let $i_{\max} = \max_{n<k\leq K}(i_k)$. Then $A_k^\pi \cdot \vec{w}_{i_{\max}} = A_k^\pi \cdot \vec{w}_{i_{\max}+j}$ for any positive integer j, and all $n < k \leq K$. Therefore since the m vectors A_k^π for $k > n$ are linearly independent, $w_{i_{\max}} = w_{i_{\max}+j}$ and the sequence stabilizes. Note that two words representing the same coset can always be compared under $\leq_\pi$, since Corollary 3.2.4 implies there are integers $a_1, \ldots, a_f$ satisfying the definition. Thus there is a unique $\leq_\pi$-minimal word in W^π that represents each coset in $(F\backslash G)^\pi$.

Note that if $v \leq_\pi w$ then $A_{n+1}^\pi \cdot \vec{v} \leq A_{n+1}^\pi \cdot \vec{w}$ and so $\omega(v) \leq \omega(w)$. Thus the unique $\leq_\pi$-minimal element in W^π that represents a given F-coset is also a minimal weight coset representative (amongst W^π). Let V_F^π denote the set of all $\leq_\pi$-minimal representatives in W^π, that is

$$V_F^\pi := \{w \in W^\pi \,|\, \text{If } v \in W^\pi \text{ s.t. } \overline{v} \in F\overline{w} \text{ then } w \leq_\pi v\}\,. \tag{3.10}$$

To finish the proof, we need to show that V_F^π corresponds to a polyhedral set in $\mathbb{Z}^m$.

An element $\tau \in \mathbb{Z}^m$ will be called a *translation* (with respect to F, π) if there exist integers $a_1, \ldots, a_f$ and $0 \leq i \leq K - n$ such that $A_k^\pi \cdot \tau = e_k \cdot \sum_{j=1}^f a_j \boldsymbol{b}_j$ for $1 \leq k \leq n$ and

$$A_{n+1}^\pi \cdot \tau = \cdots = A_{n+i-1}^\pi \cdot \tau = 0, \;\; A_{n+i}^\pi \cdot \tau > 0.$$

Let $\mathcal{T}$ denote the set of all such translations. Suppose $v, w \in W^\pi$. Then it is clear that $\overline{w} \in F\overline{v}$ with $w \leq_\pi v$ if and only if there exists some $\tau \in \mathcal{T}$ with $\vec{v} = \vec{w} + \tau$. The set $\mathcal{T}$ is contained in $\mathbb{Z}^m$. Consider $\mathcal{T} \cap Q_I$ for some $I \subseteq \{1, \ldots, m\}$. By Lemma 3.1.26, there exists a finite set $\mathcal{T}_I \subset \mathcal{T} \cap Q_I$ such that each element $\tau \in \mathcal{T} \cap Q_I$ has a bound $\tau_0 \in \mathcal{T}_I$ such that $\tau_0 \leq_I \tau$. Let $\mathcal{T}_{\min} = \bigcup_I \mathcal{T}_I$, the union of the minimal translations across all orthants. We now claim that

$$\overrightarrow{V_F^\pi} = \mathbb{N}^m \setminus \bigcup_{\tau \in \mathcal{T}_{\min}} (\tau + \mathbb{N}^m).$$

It is not hard to see that this is a polyhedral set, which proves the Proposition.

To see the claim, first suppose that $v \in V_F^\pi$ but $\vec{v} \notin \mathbb{N}^m \setminus \bigcup_{\tau \in \mathcal{T}_{\min}} (\tau + \mathbb{N}^m)$. So $\vec{v} \in \tau + \mathbb{N}^m$ for some $\tau \in \mathcal{T}_{\min}$, i.e. $\vec{v} = \tau + \vec{w}$ for some $\vec{w} \in \mathbb{N}^m$. This implies that there is some $w \in W^\pi$ which shares an F-coset with v such that $w \leq_\pi v$. But this implies that v is not minimal, contradicting the assumption that $v \in V_F^\pi$.

Conversely, suppose that $\vec{v} \in \mathbb{N}^m \setminus \bigcup_{\tau \in \mathcal{T}_{\min}} (\tau + \mathbb{N}^m)$ and $v \notin V_F^\pi$. So there exists some $\tau \in \mathcal{T}$ and $\vec{v}_0 \in V_F^\pi$, with $\vec{v} = \vec{v}_0 + \tau$ and $\overline{v_0} \in F\overline{v}$. In other words $\vec{v} - \vec{v}_0$ is a translation. Choose I so that $\vec{v} - \vec{v}_0 \in Q_I$, and then $\tau_0 \in \mathcal{T}_I$ so that $\tau_0 \leq \vec{v} - \vec{v}_0$. We claim that $\vec{v} - \tau_0 \in \mathbb{N}^m$, i.e. $\vec{v} \in \tau_0 + \mathbb{N}^m$, contradicting our assumption. Indeed, for $i \in I$ we have $e_i \cdot \tau_0 \leq e_i \cdot \tau$, so $e_i \cdot (\vec{v} - \tau_0) \geq e_i \cdot (\vec{v} - \tau) = e_i \cdot \vec{v}_0 \geq 0$, and for $i \notin I$ we have $e_i \cdot (\vec{v} - \tau_0) \geq e_i \cdot (-\tau_0) \geq 0$. So for all i, $e_i \cdot (\vec{v} - \tau_0) \geq 0$, and hence $\vec{v} - \tau \in \mathbb{N}^m$ as claimed. □

Since $Fg \subset \mathbb{Z}^n g$ for any $g \in G$, any two words that represent the same F-coset must lie in the same $\mathbb{Z}^n$-coset. So we consider each $\mathbb{Z}^n$-coset separately. Section 3.1 tells us that for a given $\mathbb{Z}^n$-coset, say $\mathbb{Z}^n t$ for $t \in T$, there is a finite set of patterns P_t, over the extended generating set $\widetilde{S}$, whose patterned sets contain representatives for all the elements of the coset (and no other cosets). We take the corresponding polyhedral sets $\overrightarrow{V_F^\pi}$ from equation (3.10) for each $\pi \in P_t$, and combine them to find a language of representatives for the F-cosets within $\mathbb{Z}^n t$. We may have pairs of words with different patterns that both represent the same coset, but we only wish to count the minimal one. To prove Theorem 3.2.2, we show that these overlaps between the $\overrightarrow{V_F^\pi}$s are polyhedral, so can be removed without losing rationality.

Definition 3.2.6. Let $\pi, \mu \in P_t$ for some $t \in T$. Define the set $R^{\pi,\mu}$ (resp. $R_*^{\pi,\mu}$) consisting of all those elements of V_F^π where there is an element of V_F^μ of strictly smaller (resp. equal) weight that represents the same coset:

$$R^{\pi,\mu} := \{v \in V_F^\pi \mid \exists u \in V_F^\mu,\ \overline{u} \in F\overline{v},\ \omega(u) < \omega(v)\}$$

$$R_*^{\pi,\mu} := \{v \in V_F^\pi \mid \exists u \in V_F^\mu,\ \overline{u} \in F\overline{v},\ \omega(u) = \omega(v)\}.$$

We need to discard all of $R^{\pi,\mu}$ for every pair $\pi \neq \mu$, since we only want minimal words. If there exist two or more minimal weight representatives for the same coset with equal weight, we must choose exactly one and discard the rest. We make the following definition.

Definition 3.2.7. Pick a total order on the finite set P_t, denoted $\pi_1 < \pi_2 < \cdots$. Let

$$U_F^{\pi_k} := V_F^{\pi_k} \setminus \left[\bigcup_{i \neq k} R^{\pi_k, \pi_i} \cup \bigcup_{j < k} R_*^{\pi_k, \pi_j} \right].$$

So $U_F^{\pi_k}$ consists of those minimal-weight coset representatives in W^{π_k} where there are no representatives of the same coset with smaller weight and a different pattern, and wherever there are multiple representatives with equal weight we choose based on the order on P_t.

Proof of Theorem 3.2.2. We claim that $\overrightarrow{U_F^{\pi_k}} \subset \mathbb{N}^{m(\pi)}$ is a polyhedral set for each π_k. Then the disjoint union of the sets $U_F^{\pi_k}$ for each $\pi_k \in P_t$, and each $t \in T$, is a finite disjoint union of rationally growing languages, forming a set of minimal weight representatives for the cosets $F \backslash G$, which will prove the Theorem.

To prove that $\overrightarrow{U_F^{\pi_k}}$ is polyhedral, it is enough to show that $\overrightarrow{R^{\pi,\mu}}$ and $\overrightarrow{R_*^{\pi,\mu}}$ are polyhedral for any π, μ, since $\overrightarrow{U_F^{\pi_k}}$ is then obtained from polyhedral sets via finite unions and set complement, so is itself polyhedral.

Let $\mathbf{1}_j \in \mathbb{Z}^{2n+2+f}$ be the vector with a 1 at the jth entry and zeroes everywhere else. Define the vectors

$$E_i = \mathbf{1}_i - \mathbf{1}_{i+n+1} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -e_i \cdot \boldsymbol{b}_1 \\ -e_i \cdot \boldsymbol{b}_2 \\ \vdots \\ -e_i \cdot \boldsymbol{b}_f \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} 2n+2 \text{ zeroes} \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} f \text{ rows} \end{matrix}$$

for each $1 \leq i \leq n$, and let $E_{n+1} = \mathbf{1}_{n+1} - \mathbf{1}_{2n+2}$. Then define the polyhedral sets

$$\Phi := \bigcap_{i=1}^{n} \left\{ \phi \in \mathbb{Z}^{2n+2+f} \mid \phi \cdot E_i = 0 \right\} \cap \left\{ \phi \in \mathbb{Z}^{2n+2+f} \mid \phi \cdot E_{n+1} > 0 \right\}$$

and

$$\Phi_* := \bigcap_{i=1}^{n} \left\{\phi \in \mathbb{Z}^{2n+2+f} \mid \phi \cdot E_i = 0\right\} \cap \left\{\phi \in \mathbb{Z}^{2n+2+f} \mid \phi \cdot E_{n+1} = 0\right\}.$$

Let $\mathcal{E}^\pi \colon \overrightarrow{V_F^\pi} \to \mathbb{Z}^{n+1}$ denote the integral affine transformation

$$\mathcal{E}^\pi \colon \vec{w} \mapsto \begin{pmatrix} A_1^\pi \cdot \vec{w} + B_1^\pi \\ A_2^\pi \cdot \vec{w} + B_2^\pi \\ \vdots \\ A_{n+1}^\pi \cdot \vec{w} + B_{n+1}^\pi \end{pmatrix}$$

for any pattern π (and write $(\mathcal{E}^\pi)^{-1}X$ for the preimage in $\overrightarrow{V_F^\pi}$ of any $X \subseteq \mathbb{Z}^{n+1}$). For any $k' > k$, write $p_k \colon \mathbb{Z}^{k'} \to \mathbb{Z}^k$ for the projection onto the first k coordinates. We will show that

$$\overrightarrow{R^{\pi,\mu}} = (\mathcal{E}^\pi)^{-1} p_{n+1}\left[\left(\mathcal{E}^\pi(\overrightarrow{V_F^\pi}) \times \mathcal{E}^\mu(\overrightarrow{V_F^\mu})\right) \cap p_{2n+2}(\Phi)\right], \tag{3.11}$$

which is a polyhedral set since projection is an affine transformation. Indeed, suppose that $v \in R^{\pi,\mu}$. So there exists $u \in V_F^\mu$ such that $\overline{u} \in F\overline{v}$ and $\omega(u) < \omega(v)$. By Corollary 3.2.4, there exist integers $a_1, \ldots, a_f$ such that

$$A_i^\pi \cdot \vec{v} + B_i^\pi - (A_i^\mu \cdot \vec{u} + B_i^\mu) = e_i \cdot \sum_{j=1}^{f} a_j \boldsymbol{b}_j$$

for each $1 \leq i \leq n$, and $A^{\pi}_{n+1} \cdot \vec{v} + B^{\pi}_{n+1} > A^{\mu}_{n+1} + B^{\mu}_{n+1}$. In other words

$$\begin{pmatrix} A^{\pi}_1 \cdot \vec{v} + B^{\pi}_1 \\ A^{\pi}_2 \cdot \vec{v} + B^{\pi}_2 \\ \vdots \\ A^{\pi}_{n+1} \cdot \vec{v} + B^{\pi}_{n+1} \\ A^{\mu}_1 \cdot \vec{u} + B^{\mu}_1 \\ A^{\mu}_2 \cdot \vec{u} + B^{\mu}_2 \\ \vdots \\ A^{\mu}_{n+1} \cdot \vec{u} + B^{\mu}_{n+1} \\ a_1 \\ \vdots \\ a_f \end{pmatrix} \in \Phi,$$

and hence $(\mathcal{E}^{\pi}(\vec{v}), \mathcal{E}^{\mu}(\vec{u})) \in p_{2n+2}(\Phi)$, so $\vec{v}$ is contained in the right hand side of (3.11).

Conversely, let $\vec{v} \in \mathbb{N}^{m(\pi)}$ be contained in the right hand side of (3.11). Thus

$$\mathcal{E}^{\pi}(\vec{v}) \in p_{n+1}\left[\left(\mathcal{E}^{\pi}(\overrightarrow{V^{\pi}_F}) \times \mathcal{E}^{\mu}(\overrightarrow{V^{\mu}_H})\right) \cap p_{2n+2}(\Phi)\right]$$

so there exists $\boldsymbol{z} \in \mathbb{Z}^{n+1}$ with

$$(\mathcal{E}^{\pi}(\vec{v}), \boldsymbol{z}) \in \left(\mathcal{E}^{\pi}(\overrightarrow{V^{\pi}_F}) \times \mathcal{E}^{\mu}(\overrightarrow{V^{\mu}_F})\right) \cap p_{2n+2}(\Phi).$$

That is, there exists $u \in V^{\mu}_F$ with $\boldsymbol{z} = (A^{\mu}_1 \cdot \vec{u} + B^{\mu}_1, \ldots, A^{\mu}_{n+1} \cdot \vec{u} + B^{\mu}_{n+1})^{\top}$ and there exist integers $a_1, \ldots, a_f$ such that $(\mathcal{E}^{\pi}(\vec{v}), \boldsymbol{z}, a_1, \ldots, a_f)^{\top} \in \Phi$, and together this means that

$$\begin{pmatrix} A_1^\pi \cdot \vec{v} + B_1^\pi \\ A_2^\pi \cdot \vec{v} + B_2^\pi \\ \vdots \\ A_{n+1}^\pi \cdot \vec{v} + B_{n+1}^\pi \\ A_1^\mu \cdot \vec{u} + B_1^\mu \\ A_2^\mu \cdot \vec{u} + B_2^\mu \\ \vdots \\ A_{n+1}^\mu \cdot \vec{u} + B_{n+1}^\mu \\ a_1 \\ \vdots \\ a_f \end{pmatrix} \in \Phi.$$

From the definition of Φ, this implies that

$$A_i^\pi \cdot \vec{v} + B_i^\pi - (A_i^\mu \cdot \vec{u} + B_i^\mu) = e_i \cdot \sum_{j=1}^{f} a_j \boldsymbol{b}_j$$

for each $1 \leq i \leq n$ and $A_{n+1}^\pi \cdot \vec{v} + B_{n+1}^\pi > A_{n+1}^\mu \cdot \vec{u} + B_{n+1}^\mu$. So $v \in V_F^\pi$ and there exists $u \in V_F^\mu$ with $\overline{u} \in F\overline{v}$ and $\omega(u) < \omega(v)$, i.e. $v \in R^{\pi,\mu}$, and so $\overrightarrow{R^{\pi,\mu}}$ has the polyhedral form (3.11) as claimed.

In an exactly analogous way,

$$\overrightarrow{R_*^{\pi,\mu}} = (\mathcal{E}^\pi)^{-1} p_{n+1} \left[\left(\mathcal{E}^\pi(\overrightarrow{V_F^\pi}) \times \mathcal{E}^\mu(\overrightarrow{V_F^\mu}) \right) \cap p_{2n+2}(\Phi_*) \right].$$

□

We now use the previous result to prove Theorem 3.2.1, that is, to show that for an arbitrary subgroup $H \leq G$, the set of right cosets $H\backslash G$ has rational growth with respect to any choice of weighted generating set for G. The proof relies on the understanding of the structure of subgroups in Lemma 3.4.4 and the rationality of coset growth for free abelian subgroups in Theorem 3.2.2.

Proof of Theorem 3.2.1. First, we consider the coset structure of G. Let $\mathbb{Z}^n$ be the maximal free abelian normal subgroup of G. Then we have $H \cap \mathbb{Z}^n \lhd H$, with finite index $c \leq d$. Fix a choice of transversal $\{h_1, \ldots, h_c\}$ for the cosets $(H \cap \mathbb{Z}^n)\backslash H$. As in Lemma 3.4.4, we extend this to a transversal for $\mathbb{Z}^n\backslash G$, write $T = \{t_1, \ldots, t_d\} \supseteq \{h_1, \ldots, h_c\}$. Suppose that the free abelian group $H \cap \mathbb{Z}^n$ has rank f, and fix a basis $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_f\} \subset \mathbb{Z}^n$.

Consider a coset $Hg \in H\backslash G$. Following the above discussion, we may decompose H as the finite union of its $H \cap \mathbb{Z}^n$-cosets. We may also write $g = (g_1, \ldots, g_n)^\top t$ for some $g_i \in \mathbb{Z}$ and $t \in T$. Therefore

$$Hg = \left(\bigcup_{j=1}^{c} (H \cap \mathbb{Z}^n) h_j\right) \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} t.$$

Since $H \cap \mathbb{Z}^n$ is also a subgroup of G, contained in $\mathbb{Z}^n$, Theorem 3.2.2 provides a minimal weight representative for each coset of $H \cap \mathbb{Z}^n$ in G of the form $(H \cap \mathbb{Z}^n)h_j g$, and therefore a collection of c candidates for a minimal-weight representative for Hg, since the minimal-weight representative for Hg is one of the c minimal-weight representatives for the cosets $(H \cap \mathbb{Z}^n)h_j g$. We will express these candidates as c-tuples of words, (whose patterns together make c-fold patterns, see Section 3.1.5), and show that they correspond to polyhedral sets, from which rationality will follow.

Fix a c-fold pattern $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_c) \in P^c$ (recalling the definition of P from Definition 3.1.7), and define

$$\begin{aligned} V(\boldsymbol{\pi}) = \Big\{ \big(w^{(1)}, \ldots, w^{(c)}\big) \in U^{\pi_1}_{H\cap\mathbb{Z}^n} \times \cdots \times U^{\pi_c}_{H\cap\mathbb{Z}^n} \mid \\ \exists g \in G \text{ s.t. } \overline{w^{(j)}} \in (H \cap \mathbb{Z}^n)h_j g,\ 1 \le j \le c \Big\}, \end{aligned} \tag{3.12}$$

where $U^{\pi_k}_{H\cap\mathbb{Z}^n}$ is the set of minimal representatives for the cosets of $H \cap \mathbb{Z}^n$ in G as defined in Definition 3.2.7, where $H \cap \mathbb{Z}^n$ plays the role of F. Each element of $V(\boldsymbol{\pi})$ consists of a c-tuple of candidates for a minimal weight representative of some coset Hg. By Lemma 3.1.24, if $\overrightarrow{V(\boldsymbol{\pi})}$ is polyhedral then there is a language $\mathcal{L}_{\boldsymbol{\pi}}$ of minimal representatives for those cosets represented by $V(\boldsymbol{\pi})$, which grows rationally. Every element of G has a minimal-weight representative with a pattern in P (by definition of P), and so in particular every coset has a minimal-weight representative with pattern in P, and is therefore represented in some $V(\boldsymbol{\pi})$. Thus the union $\bigcup_{\boldsymbol{\pi}\in P^c} \mathcal{L}_{\boldsymbol{\pi}}$ forms a language of minimal weight representatives for the set of cosets $H\backslash G$. Since

P^c is finite, this union is a finite union of polyhedral sets, and thus has rational growth.

We now show that $\overrightarrow{V(\boldsymbol{\pi})}$ is indeed a polyhedral set for each $\boldsymbol{\pi} \in P^c$, which will complete the proof. For some tuple in $V(\boldsymbol{\pi})$, consider the element g as in equation (3.12). This can be expressed as $(g_1, \ldots, g_n)^\top t$ for some $g_i \in \mathbb{Z}$ and $t \in T$.

So we can decompose $V(\boldsymbol{\pi})$ as a finite union $\bigcup_{t \in T} V(\boldsymbol{\pi}, t)$ where

$$V(\boldsymbol{\pi}, t) = \Big\{ \big(w^{(1)}, \ldots, w^{(c)} \big) \in U^{\pi_1}_{H \cap \mathbb{Z}^n} \times \cdots \times U^{\pi_c}_{H \cap \mathbb{Z}^n} \mid \\ \exists g \in \mathbb{Z}^n t \text{ s.t. } \overline{w^{(j)}} \in (H \cap \mathbb{Z}^n) h_j g,\ 1 \leq j \leq c \Big\}.$$

We wish to write an element of $(H \cap \mathbb{Z}^n) h_j (g_1, \ldots, g_n)^\top t$ in the standard form defined in section 3.1.1. Recall from section 3.1.3 that for any element $s \in T$, we have a matrix Δ_s so that $s(a_1, \ldots, a_n)^\top s^{-1} = \Delta_s (a_1, \ldots, a_n)^\top$ for any integers a_i. For two coset representatives $s, t \in T$, their product st will not necessarily be in T. So let $x_{st} \in \mathbb{Z}^n$ and $\tau_{st} \in T$ be such that $st = x_{st} \tau_{st}$.

Now suppose that

$$\gamma \in (H \cap \mathbb{Z}^n) h_j \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} t \subseteq H \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} t.$$

So there exist integers $\lambda_1^{(j)}, \ldots, \lambda_f^{(j)}$, so that

$$\gamma = \left(\sum_{k=1}^{f} \lambda_k^{(j)} \boldsymbol{b}_k\right) h_j \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} t$$

$$= \left(\sum_{k=1}^{f} \lambda_k^{(j)} \boldsymbol{b}_k + \Delta_{h_j} \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} \right) h_j t$$

$$= \left(\sum_{k=1}^{f} \lambda_k^{(j)} \boldsymbol{b}_k + \Delta_{h_j} \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} + x_{h_j t} \right) \tau_{h_j t},$$

which is in the standard form. Thus for $w^{(j)} \in W^{\pi_j}$, we have $\overline{w^{(j)}} \in (H \cap \mathbb{Z}^n) h_j g$ for some $g = (g_1, \ldots, g_n)^\top t$ if and only if

1. there exists $\lambda_k^{(j)} \in \mathbb{Z}$ for each $1 \leq k \leq f$ such that

$$\begin{pmatrix} A_1^{\pi_j} \cdot \overrightarrow{w^{(j)}} \\ A_2^{\pi_j} \cdot \overrightarrow{w^{(j)}} \\ \vdots \\ A_n^{\pi_j} \cdot \overrightarrow{w^{(j)}} \end{pmatrix} + \begin{pmatrix} B_1^{\pi_j} \\ B_2^{\pi_j} \\ \vdots \\ B_n^{\pi_j} \end{pmatrix} = \sum_{k=1}^{f} \lambda_k^{(j)} \boldsymbol{b}_k + \Delta_{h_j} \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} + x_{h_j t},$$

   and

2. $t_{\pi_j} = \tau_{h_j t}$,

(where $\overline{\pi_j} = (B_1^{\pi_j}, \ldots, B_n^{\pi_j})^\top t_{\pi_j}$ as before). The first condition can be restated as follows. There exists $\lambda_k^{(j)} \in \mathbb{Z}$ for each $1 \leq k \leq f$ such that

$$A_i^{\pi_j} \cdot \overrightarrow{w^{(j)}} + B_i^{\pi_j} = e_i \cdot \left(\sum_{k=1}^{f} \lambda_k^{(j)} \boldsymbol{b}_k + \Delta_{h_j} \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} + x_{h_j} t \right)$$

for each $1 \leq i \leq n$ and $1 \leq j \leq c$. This can be re-written as

$$A_i^{\pi_j} \cdot \overrightarrow{w^{(j)}} + \sum_{k=1}^{f} \lambda_k (-e_i \cdot \boldsymbol{b}_k) - e_i \cdot \Delta_{h_j} \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} = -B_i^{\pi_j} + e_i \cdot x_{h_j} t. \tag{3.13}$$

Then we may rewrite $V(\boldsymbol{\pi}, t)$ as follows.

$$V(\boldsymbol{\pi}, t) = \Big\{ \big(w^{(1)}, \ldots, w^{(c)}\big) \in U_{H \cap \mathbb{Z}^n}^{\pi_1} \times \cdots \times U_{H \cap \mathbb{Z}^n}^{\pi_c} \,\Big|\, \exists (g_1, \ldots, g_n)^\top \in \mathbb{Z}^n,$$
$$\lambda_k^{(j)} \in \mathbb{Z}^n, \text{ s.t. each } w^{(j)} \text{ satisfies (3.13) for each } 1 \leq i \leq n \Big\}$$

For a fixed $\boldsymbol{\pi}$ and $t \in T$, we define vectors $L_i^j \in \mathbb{Z}^{m(\boldsymbol{\pi})+fc+n}$ as follows:

$$L_i^j := \begin{pmatrix} 0\\ \vdots\\ 0\\ A_i^{\pi_j}\\ 0\\ \vdots\\ 0\\ 0\\ \vdots\\ 0\\ -e_i\cdot \boldsymbol{b}_1\\ \vdots\\ -e_i\cdot\boldsymbol{b}_f\\ 0\\ \vdots\\ 0\\ -\mathbf{1}_i\Delta_{h_j}\end{pmatrix}
\begin{array}{l}
\left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} \sum_{k=1}^{j-1} m(\pi_j) \text{ zeroes}\\
m(\pi_j) \text{ rows}\\
\left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} \sum_{k=j+1}^{c} m(\pi_j) \text{ zeroes}\\
\left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} f(j-1) \text{ zeroes}\\
\left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} f \text{ rows}\\
\left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} f(c-j) \text{ zeroes}\\
n \text{ rows}
\end{array}$$

where $\mathbf{1}_i\Delta_{h_j}$ is the matrix product of the row vector with 1 at the ith position and zeroes elsewhere, and Δ_{h_j}.

Now, noting that

$$\mathbf{1}_i\Delta_{h_j} \cdot \begin{pmatrix} g_1\\ g_2\\ \vdots\\ g_n\end{pmatrix} = e_i \cdot \left(\Delta_{h_j} \begin{pmatrix} g_1\\ g_2\\ \vdots\\ g_n\end{pmatrix}\right),$$

we see that a c-tuple of words $\left(w^{(1)}, \ldots, w^{(c)}\right)$ satisfies (3.13) for some i precisely when there exists $v \in \mathbb{Z}^{m(\boldsymbol{\pi})+fc+n}$ such that

$$L_i^j \cdot v = -B_i^{\pi_j} + e_i \cdot x_{h_j t}$$

and $p_{m(\boldsymbol{\pi})}(v) = \left(\overrightarrow{w^{(1)}}, \ldots, \overrightarrow{w^{(c)}}\right) \in \mathbb{Z}^{m(\boldsymbol{\pi})}$. Therefore we have

$$\overrightarrow{V(\boldsymbol{\pi}, t)} = p_{m(\boldsymbol{\pi})}\left(\bigcap_{j=1}^{c}\bigcap_{i=1}^{n}\left\{v \in \mathbb{Z}^{m(\boldsymbol{\pi})+fc+n} \,\middle|\, L_i^j \cdot v = -B_i^{\pi_j} + e_i \cdot x_{h_j t}\right\}\right),$$

which is a positive polyhedral subset of $\mathbb{Z}^{m(\boldsymbol{\pi})}$. Now since finite unions of polyhedral sets are polyhedral, $\overrightarrow{V(\boldsymbol{\pi})} = \bigcup_{t \in T} \overrightarrow{V(\boldsymbol{\pi}, t)}$ is polyhedral, which proves the Theorem.

$\square$

Benson's result that virtually abelian groups have rational weighted growth series with respect to all generating sets follows from the following special case of Theorem 3.2.1.

Theorem 3.2.8 (Theorem 1.2 of [4]). *Let G be a virtually abelian group, with any choice of finite weighted generating set S. For each pattern $\pi \in P$, with P as in definition 3.1.7, there exists a set $U^\pi \subset \widetilde{S}^*$ such that $\overrightarrow{U^\pi} \subset \mathbb{N}^{m(\pi)}$ is polyhedral, and the disjoint union $\bigcup_{\pi \in P} U^\pi$ forms a language of unique minimal-weight representatives for the elements of G.*

Since the x_i generators in U^π correspond to the coordinates of $\mathbb{N}^{m(\pi)}$, and the contribution from the y_k generators is constant within U^π, rational growth of $\overrightarrow{U^\pi}$ (in the sense of Proposition 3.1.21) implies rational growth of U^π with respect to $\widetilde{S}$, and hence with respect to S.

3.3 Conjugacy Growth Series

In this section we will prove the following.

Theorem 3.3.1. *Let G be a virtually abelian group, with finite generating set S, and weight function $\omega \colon S \to \mathbb{N}_+$. Then the weighted conjugacy growth series of G with respect to S is rational.*

In order to prove this Theorem, we show that the set of conjugacy classes of a virtually abelian group can be split into an infinite collection of finite classes, and an infinite collection of infinite classes. For the finite case, [4] gives us a way to find a minimal representative for each element of the conjugacy class, and express a full set of representatives using polyhedral sets. In the infinite case, we express each conjugacy class as a finite union of cosets of certain subgroups. Section 3.2 gives us a way to find a minimal representative word for a given coset, and to express a full set of such representatives using polyhedral sets. We thus find a finite set of candidates for a unique minimal representative for every conjugacy class (finite or infinite). This allows us to use Lemma 3.1.24 to extract a single such representative for each class, so that the polyhedral set description, and thus rational growth, is preserved.

As above, we assume that G contains $\mathbb{Z}^n$ as a normal subgroup, with $[G\colon \mathbb{Z}^n] = d < \infty$, and we let T be a choice of transversal for $G/\mathbb{Z}^n$ such that $1_G \in T$. Furthermore, we fix an order on T:

$$T = \{1, t_2, t_3, \ldots, t_d\}.$$

First, we must understand the structure of conjugacy classes in virtually abelian groups. Conjugacy classes have different structure depending on whether they are inside or outside the centralizer of $\mathbb{Z}^n$, $C_G(\mathbb{Z}^n)$. Thus we consider these cases separately. Note that if one element of a coset $\mathbb{Z}^n t$ centralizes $\mathbb{Z}^n$ then t must centralize $\mathbb{Z}^n$ and hence the whole coset is in $C_G(\mathbb{Z}^n)$. So both $C_G(\mathbb{Z}^n)$ and $G \setminus C_G(\mathbb{Z}^n)$ are unions of $\mathbb{Z}^n$-cosets.

3.3.1 Conjugacy classes of elements inside the centralizer of $\mathbb{Z}^n$

Lemma 3.3.2. *Let $g \in C_G(\mathbb{Z}^n)$. Then the conjugacy class of g has size at most d, and is given by*

$$[g] = \{tgt^{-1} \mid t \in T\} = \{g, t_2 g t_2^{-1} \ldots, t_d g t_d^{-1}\}.$$

Proof. Let $h \in G$. Then $hgh^{-1} = xtgt^{-1}x^{-1}$ for some $x \in \mathbb{Z}^n$ and $t \in T$. Since the centralizer of a normal subgroup is itself a normal subgroup, tgt^{-1} centralizes $\mathbb{Z}^n$, and so $hgh^{-1} = tgt^{-1}$. □

Recall the sets U^π of minimal-length representatives for π-patterned words, introduced in Theorem 3.2.8. Each element of a conjugacy class has a unique minimal-weight representative, and this is contained in U^π for some $\pi \in P$ (recall Definition 3.1.7). So by Lemma 3.3.2, each conjugacy class in $C_G(\mathbb{Z}^n)$ has at most d candidate words for a weight minimal representative. A d-tuple of candidates has a d-fold pattern, the d-dimensional vector where the entries are the patterns of the component words of the d-tuple. We will show that for each d-fold pattern in P^d, the corresponding set of d-tuples of candidate representatives forms a polyhedral set.

Definition 3.3.3. Fix a d-fold pattern $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_d) \in P^d$ where $\overline{\pi_j} \in C_G(\mathbb{Z}^n)$ for each j, and $t_j \pi_1 t_j^{-1} \in \mathbb{Z}^n \pi_j$ for each $2 \leq j \leq d$. Let

$$C(\boldsymbol{\pi}) := \Big\{ \big(w^{(1)}, \dots, w^{(d)}\big) \in U^{\pi_1} \times \cdots \times U^{\pi_d} \mid \qquad\qquad \overline{w^{(j)}} = t_j \overline{w^{(1)}} t_j^{-1}, \ \text{ for each } 2 \leq j \leq d \Big\}.$$

Remark 3.3.4. *Note that each tuple in $C(\boldsymbol{\pi})$ corresponds to a conjugacy class, and (by definition of the sets U^{π_j}) the weight of a conjugacy class is realised by at least one of the words in the corresponding tuple.*

Proposition 3.3.5. *For each $C(\boldsymbol{\pi})$, the set $\overrightarrow{C(\boldsymbol{\pi})} \subset \mathbb{N}^{m(\boldsymbol{\pi})}$ is polyhedral.*

Proof. Consider $w^{(1)} \in U^{\pi_1}$ and $w^{(j)} \in U^{\pi_j}$ where $t_j \overline{\pi_1} t_j^{-1} \in \mathbb{Z}^n \overline{\pi_j}$. By Lemma 3.1.17, $\overline{w^{(j)}} = t_j \overline{w^{(1)}} t_j^{-1}$ if and only if

$$A^{\pi_1}_{i,t_j} \cdot \overrightarrow{w^{(1)}} - A^{\pi_j}_i \cdot \overrightarrow{w^{(j)}} = B^{\pi_j}_i - B^{\pi_1}_{i,t_j} \tag{3.14}$$

for each $1 \leq i \leq n$. We express this using linear algebra. Define

$$F_i^j(\boldsymbol{\pi}) = \begin{pmatrix} A^{\pi_1}_{i,t_j} \\ 0 \\ \vdots \\ 0 \\ -A^{\pi_j}_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} m(\pi_1) \text{ rows} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \sum_{k=2}^{j-1} m(\pi_k) \text{ zeroes} \\ m(\pi_j) \text{ rows} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \sum_{k=j+1}^{d} m(\pi_k) \text{ zeroes} \end{array}$$

for each $1 \le i \le n$, $2 \le j \le d$. Then the vector $\left(\overrightarrow{w^{(1)}}, \dots, \overrightarrow{w^{(d)}}\right)^\top$ satisfies (3.14) for some j if and only if

$$F_i^j(\boldsymbol{\pi}) \cdot \begin{pmatrix} \overrightarrow{w^{(1)}} \\ \vdots \\ \overrightarrow{w^{(d)}} \end{pmatrix} = B_i^{\pi_j} - B_{i,t_j}^{\pi_1}$$

for each $1 \le i \le n$. Thus

$$\overrightarrow{C(\boldsymbol{\pi})} = \left(\overrightarrow{U^{\pi_1}} \times \cdots \times \overrightarrow{U^{\pi_d}}\right) \cap \bigcap_{j=2}^{d} \bigcap_{i=1}^{n} \left\{ \vec{z} \in \mathbb{Z}^{m(\boldsymbol{\pi})} \,\middle|\, F_i^j(\boldsymbol{\pi}) \cdot \vec{z} = B_i^{\pi_j} - B_{i,t_j}^{\pi_1} \right\}.$$

This is therefore a polyhedral set. □

3.3.2 Conjugacy classes of elements outside the centralizer of $\mathbb{Z}^n$

We express the conjugacy classes in terms of certain cosets, and use the sets U_F^π introduced in Definition 3.2.7 to find polyhedral sets of conjugacy class representatives.

Definition 3.3.6. For any $\gamma \in G$, define the subgroup

$$F(\gamma) = \{[x, \gamma] \mid x \in \mathbb{Z}^n\}.$$

Note that this is indeed a subgroup of G, since if $x, y \in \mathbb{Z}^n$, we have

$$[x,\gamma][y,\gamma] = x\gamma x^{-1}\gamma^{-1}y\gamma y^{-1}\gamma^{-1} = xy\gamma x^{-1}\gamma^{-1}\gamma y^{-1}\gamma^{-1} = [xy, \gamma].$$

Furthermore, since $\mathbb{Z}^n$ is normal, $[x,\gamma] \in \mathbb{Z}^n$, and so $F(\gamma)$ is a subgroup of $\mathbb{Z}^n$, and hence is free abelian.

Remark 3.3.7. *Let $a \in \mathbb{Z}^n$ and $t \in T$. Then since $\mathbb{Z}^n$ is normal, $[x, at] = xatx^{-1}t^{-1}a^{-1} = xtx^{-1}t^{-1}aa^{-1} = [x,t]$. So $F(\gamma)$ depends only on the $\mathbb{Z}^n$-coset that γ is contained in. Thus if $w \in W^\pi$ then $F(\overline{w}) = F(\overline{\pi})$.*

If A and B are subsets of some group G, write ${}^{B}A$ for the conjugate of A by B, that is ${}^{B}A = \{bab^{-1} \mid a \in A, b \in B\}$.

Lemma 3.3.8. *If $g \in G \setminus C_G(\mathbb{Z}^n)$ then its conjugacy class is given by a union of finitely many cosets as follows*

$$[g] = \bigcup_{t \in T} \left\{ [x, tgt^{-1}] \mid x \in \mathbb{Z}^n \right\} tgt^{-1} = \bigcup_{t \in T} F(tgt^{-1}) tgt^{-1}.$$

Proof. Let $g \in G \setminus C_G(\mathbb{Z}^n)$, and suppose $x \in \mathbb{Z}^n$. We have $xgx^{-1} = xgx^{-1}g^{-1}g = [x, g]g$. Now the conjugacy class is given by

$$\begin{aligned} [g] = {}^G\{g\} &= \{xtg(xt)^{-1} \mid x \in \mathbb{Z}^n, t \in T\} = {}^{\mathbb{Z}^n}\{tgt^{-1} \mid t \in T\} \\ &= \bigcup_{t \in T} {}^{\mathbb{Z}^n}\{tgt^{-1}\} = \bigcup_{t \in T} \left\{ \left[x, tgt^{-1}\right] \mid x \in \mathbb{Z}^n \right\} tgt^{-1}. \end{aligned}$$

□

Definition 3.3.9. Fix a d-fold pattern $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_d)$ where $\overline{\pi_j} \in G \setminus C_G(\mathbb{Z}^n)$ for each j, and $t_j \overline{\pi_1} t_j^{-1} \in \mathbb{Z}^n \overline{\pi_j}$ for $2 \leq j \leq d$. Define

$$C'(\boldsymbol{\pi}) = \left\{ \left(w^{(1)}, \ldots, w^{(d)}\right) \in U^{\pi_1}_{F(\overline{\pi_1})} \times \cdots \times U^{\pi_d}_{F(\overline{\pi_d})} \;\middle|\; \overline{w^{(j)}} \in {}^{\mathbb{Z}^n t_j}\left\{\overline{w^{(1)}}\right\},\ 2 \leq j \leq d \right\},$$

where $U^{\pi_i}_{F(\overline{\pi_i})}$ is as in Definition 3.2.7. That is, $C'(\boldsymbol{\pi})$ is the set of d-tuples of words where each $w^{(j)}$ is the unique minimal representative for its $F(\overline{\pi_j})$-coset, and $\overline{w^{(j)}}$ is conjugate to $\overline{w^{(1)}}$ via an element of $\mathbb{Z}^n t_j$.

Proposition 3.3.10. *Each element of $C'(\boldsymbol{\pi})$ consists of a d-tuple of words representing elements of the same conjugacy class. Furthermore, the weight of the conjugacy class is realised by the word(s) of smallest weight in the d-tuple.*

Proof. Let $\left(w^{(1)}, w^{(2)}, \ldots, w^{(d)}\right) \in C'(\boldsymbol{\pi})$. From the definition of $C'(\boldsymbol{\pi})$, each $\overline{w^{(j)}}$ is conjugate to $\overline{w^{(1)}}$, so each component word represents an element of the same conjugacy class. Now from Lemma 3.3.8, we see that each word represents one of the finite number of cosets that make up the corresponding conjugacy class. In fact, since each $w^{(j)}$ is contained in $U_{F(\overline{\pi_j})}$, each component word is a minimal-weight representative for the coset. Therefore the minimal-weight representative(s) for the conjugacy class must be contained in $\{w^{(1)}, w^{(2)}, \ldots, w^{(d)}\}$. □

Proposition 3.3.11. *The set* $\overrightarrow{C'(\boldsymbol{\pi})} \in \mathbb{N}^{m(\boldsymbol{\pi})}$ *is a polyhedral set.*

Proof. We have

$$^{\mathbb{Z}^n t_j}\left\{\overline{w^{(1)}}\right\} = F\left(t_j\overline{w^{(1)}}t_j^{-1}\right)t_j\overline{w^{(1)}}t_j^{-1} = F(\overline{\pi_j})t_j\overline{w^{(1)}}t_j^{-1}.$$

by Remark 3.3.7. Thus $\overline{w^{(j)}} \in {}^{\mathbb{Z}^n t_j}\left\{\overline{w^{(1)}}\right\}$ if and only if there exists $y \in F(\overline{\pi_j})$ with

$$\overline{w^{(j)}} = yt_j\overline{w^{(1)}}t_j^{-1}. \tag{3.15}$$

Write f_j for the rank of the free abelian group $F(\overline{\pi_j}) \subset \mathbb{Z}^n$. Let $\{\boldsymbol{b}_1^{\pi_j}, \ldots, \boldsymbol{b}_{f_j}^{\pi_j}\}$ be a choice of basis for $F(\overline{\pi_j})$. Then there exists $y \in F(\overline{\pi_j})$ satisfying (3.15) if and only if there exist integers $a_1, \ldots, a_{f_j}$ with

$$\overline{w^{(j)}} = \sum_{k=1}^{f_j} a_k\boldsymbol{b}_k^{\pi_j}t_j\overline{w^{(1)}}t_j^{-1}.$$

Expanding using identities (3.6) and (3.9) gives

$$\left[\begin{pmatrix} A_1^{\pi_j}\cdot\overrightarrow{w^{(j)}} \\ A_2^{\pi_j}\cdot\overrightarrow{w^{(j)}} \\ \vdots \\ A_n^{\pi_j}\cdot\overrightarrow{w^{(j)}}\end{pmatrix} + \begin{pmatrix} B_1^{\pi_j} \\ B_2^{\pi_j} \\ \vdots \\ B_n^{\pi_j}\end{pmatrix}\right]t_{\pi_j} = \left(\sum_{k=1}^{f_j} a_k\boldsymbol{b}_k^{\pi_j}\right)\left[\begin{pmatrix} A_{1,t_j}^{\pi_1}\cdot\overrightarrow{w^{(1)}} \\ A_{2,t_j}^{\pi_1}\cdot\overrightarrow{w^{(1)}} \\ \vdots \\ A_{n,t_j}^{\pi_1}\cdot\overrightarrow{w^{(1)}}\end{pmatrix} + \begin{pmatrix} B_{1,t_j}^{\pi_1} \\ B_{2,t_j}^{\pi_1} \\ \vdots \\ B_{n,t_j}^{\pi_1}\end{pmatrix}\right]t_{\pi_j},$$

or equivalently,

$$A_{i,t_j}^{\pi_1}\cdot\overrightarrow{w^{(1)}} - A_i^{\pi_j}\cdot\overrightarrow{w^{(j)}} + e_i\cdot\sum_{k=1}^{f_j} a_k\boldsymbol{b}_k^{\pi_j} = B_i^{\pi_j} - B_{i,t_j}^{\pi_1} \tag{3.16}$$

for each $1 \le i \le n$. We express this using linear algebra.

For each $1 \leq i \leq n$ and $2 \leq j \leq d$, consider the vectors

$$M_i^j(\boldsymbol{\pi}) = \begin{pmatrix} A_{i,t_j}^{\pi_1} \\ 0 \\ \vdots \\ 0 \\ -A_i^{\pi_j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} m(\pi_1) \text{ rows} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \sum_{k=2}^{j-1} m(\pi_k) \text{ zeroes} \\ m(\pi_j) \text{ rows} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \sum_{k=j+1}^{d} m(\pi_k) \text{ zeroes} \end{array}$$

and

$$N_i^j(\boldsymbol{\pi}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e_i \cdot \boldsymbol{b}_1^{\pi_j} \\ e_i \cdot \boldsymbol{b}_2^{\pi_j} \\ \vdots \\ e_i \cdot \boldsymbol{b}_{f_j}^{\pi_j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \sum_{k=2}^{j-1} f_k \text{ zeroes} \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} f_j \text{ rows} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \sum_{k=j+1}^{d} f_k \text{ zeroes} \end{array}$$

Let $f = \sum_{j=1}^{d} f_j$, i.e. the sum of the ranks of the free abelian subgroups $F(\overline{\pi_j})$, and hence the dimension of $N_i^j(\boldsymbol{\pi})$. Now by equation (3.16), $\overline{w^{(j)}} \in {}^{\mathbb{Z}^n t_j}\left\{\overline{w^{(1)}}\right\}$ precisely when there exist integers $a_1, a_2, \ldots a_f$ such that

$$M_i^j(\boldsymbol{\pi}) \cdot \begin{pmatrix} \overrightarrow{w^{(1)}} \\ \overrightarrow{w^{(2)}} \\ \vdots \\ \overrightarrow{w^{(d)}} \end{pmatrix} + N_i^j(\boldsymbol{\pi}) \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_f \end{pmatrix} = B_i^{\pi_j} - B_{i,t_j}^{\pi_1}$$

for each $1 \leq i \leq n$ and $2 \leq j \leq d$. We see that each of these identities defines an elementary set if we rewrite it as follows:

$$\left\{ \vec{z} \in \mathbb{N}^{m(\boldsymbol{\pi})+f} \;\middle|\; \begin{pmatrix} | \\ M_i^j(\boldsymbol{\pi}) \\ | \\ | \\ N_i^j(\boldsymbol{\pi}) \\ | \end{pmatrix} \cdot \vec{z} = B_i^{\pi_j} - B_{i,t_j}^{\pi_1} \right\}.$$

Taking the intersection of each elementary set, discarding the vector $(a_1, \ldots, a_r)$, and intersecting with the cartesian product of polyhedral sets $U_{F(\overline{\pi_1})} \times \cdots \times U_{F(\overline{\pi_d})}$, allows us to express the $m(\boldsymbol{\pi})$-dimensional vectors corresponding to $C'(\boldsymbol{\pi})$ as a polyhedral set:

$$\overrightarrow{C'(\boldsymbol{\pi})} = \left(U_{F(\overline{\pi_1})} \times \cdots \times U_{F(\overline{\pi_d})}\right) \cap$$

$$\bigcap_{j=2}^{d} \bigcap_{i=1}^{n} p_{m(\boldsymbol{\pi})} \left\{ \vec{z} \in \mathbb{N}^{m(\boldsymbol{\pi})+r} \;\middle|\; \begin{pmatrix} | \\ M_i^j(\boldsymbol{\pi}) \\ | \\ | \\ N_i^j(\boldsymbol{\pi}) \\ | \end{pmatrix} \cdot \vec{z} = B_i^{\pi_j} - B_{i,t_j}^{\pi_1} \right\}$$

where, as before, p_k denotes projection onto the first k coordinates. □

We are now ready to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. Each conjugacy class in G has a d-tuple of candidates for a weight-minimal representative (see Remark 3.3.4 and Proposition 3.3.10), given by an element of $C(\boldsymbol{\pi})$ or $C'(\boldsymbol{\pi})$ for an appropriate d-fold pattern $\boldsymbol{\pi}$. The result will follow from the following claim:

There exists a finite set of d-fold patterns, R, so that:

1. the candidate representatives for every conjugacy class in G (as d-tuples of words) have a d-fold pattern in R, and
2. no conjugacy class is represented by more than one d-fold pattern in R.

For each $\boldsymbol{\pi} \in R$, Lemma 3.1.24 yields a set $\mathcal{L}_{\boldsymbol{\pi}} \in \widetilde{S}^*$ of unique, minimal-weight representatives for the tuples of $C(\boldsymbol{\pi})$, or $C'(\boldsymbol{\pi})$. It follows from the above claim that $\bigcup_{\boldsymbol{\pi}\in R} \mathcal{L}_{\boldsymbol{\pi}}$ is a finite disjoint union of sets, forming a language of unique minimal-weight representatives for the conjugacy classes of G. Since each $\mathcal{L}_{\boldsymbol{\pi}}$ has rational growth series, we conclude that G has rational conjugacy growth series.

Now we prove the claim. If P is the finite set of patterns (with respect to $\widetilde{S}^*$) providing minimal weight representatives for each element of G (as per Definition 3.1.7), consider the set of ordered d-tuples $(\pi_1, \pi_2, \ldots, \pi_d)$ of elements of P, with the condition that $t_i \pi_1 t_i^{-1} \in \mathbb{Z}^n \pi_j$. For any set of such d-tuples which are permutations of each other, choose only one (arbitrarily), and discard the others. Call the resulting reduced set of d-fold patterns R. This is clearly a finite set, and is sufficient to represent all d-tuples of elements of G. This proves part (1).

To see part (2), note that the candidates are uniquely determined (either the unique weight-minimal representatives for each element, in the $C_G(\mathbb{Z}^n)$ case, or the unique weight-minimal representatives for each coset component, in the $G \setminus C_G(\mathbb{Z}^n)$ case). A tuple of candidates uniquely determines a d-fold pattern in R (since we have removed permutations). Thus the claim holds, and the theorem follows. $\square$

3.4 Relative growth series

Given a group G with finite generating set S, consider a subset $V \subset G$. One can construct the relative (weighted) growth series of V with respect to the generators of G by defining $\sigma(n)$ to be the number of elements of V with weight n. We prove that if V can be represented by polyhedral sets in an appropriate way then the relative growth of V is rational. In particular, we prove that for any *subgroup* of a virtually abelian group, the relative weighted growth series is rational (for any finite generating set and positive integer weight function).

We will use the polyhedral sets U^π from Benson's Theorem 3.2.8, which together are in one-to-one correspondence with a language of geodesic representatives for the elements of G.

Theorem 3.4.1. *Let G be virtually abelian, with normal free abelian subgroup $\mathbb{Z}^n$, and corresponding transversal T, and let S be a finite weighted generating set. Let*

$V \subseteq G$ such that there exist polyhedral subsets $V_t \subseteq \mathbb{Z}^n$ with $V = \bigcup_{t \in T} V_t t$. Then V has rational weighted growth series with respect to S.

Proof. Let P_t denote the (finite) set of patterns π such that $\overline{\pi} \in \mathbb{Z}^n t$ and $|\pi| \leq [G\colon \mathbb{Z}^n]$ (and so every element of $\mathbb{Z}^n t$ has a geodesic representative in some polyhedral set $\overrightarrow{U^\pi}$).

Let

$$\begin{aligned} R_\pi &= \{u \in \overrightarrow{U^\pi} \mid \mathcal{E}^\pi(u) \in V_t\} \\ &= (\mathcal{E}^\pi)^{-1}\left[\mathcal{E}^\pi(\overrightarrow{U^\pi}) \cap V_t\right]. \end{aligned}$$

So R_π is a polyhedral subset of $\mathbb{Z}^{m(\pi)}$, and thus $\sum_{n \geq 0} \sigma_{R_\pi}(n) z^n \in \mathbb{Q}(x)$. We have

$$\begin{aligned} \sum_{n \geq 0} \sigma_V(n) z^n &= \sum_{\pi \in P} \sum_{n \geq 0} \sigma_{R_\pi}(n - \omega(\pi)) z^n \\ &= \sum_{\pi \in P} \sum_{n \geq 0} \sigma_{R_\pi}(n) z^{n + \omega(n)} \\ &= \sum_{\pi \in P} z^{\omega(\pi)} \sum_{n \geq 0} \sigma_{R_\pi}(n) z^n. \end{aligned}$$

Since P is finite, this is a finite sum of rational functions, and thus V has rational relative growth series. □

We now show that we can test for subgroup membership using polyhedral sets, resulting in the following.

Theorem 3.4.2. *Let G be a finitely generated virtually abelian group and let H be any subgroup of G. Then H has rational weighted growth series relative to any choice of generators of G (with any weight function).*

We need the following Lemmas.

Lemma 3.4.3. *Any subgroup of a virtually abelian group is virtually abelian.*

Proof. Let G be virtually abelian, with normal free abelian finite index subgroup $\mathbb{Z}^n$ as usual. Let $d = [G\colon \mathbb{Z}^n]$. Suppose H is a proper subgroup of G. By the second isomorphism theorem, we have $H\mathbb{Z}^n \leq G$, $H \cap \mathbb{Z}^n \lhd H$, and

$$\frac{H}{H \cap \mathbb{Z}^n} \cong \frac{H\mathbb{Z}^n}{\mathbb{Z}^n}.$$

Since $[G\colon \mathbb{Z}^n] = [G\colon H\mathbb{Z}^n][H\mathbb{Z}^n\colon \mathbb{Z}^n]$ and $[G\colon \mathbb{Z}^n]$ is finite, $[H\mathbb{Z}^n\colon \mathbb{Z}^n]$ is also finite (with $[H\mathbb{Z}^n\colon \mathbb{Z}^n] \leq [G\colon \mathbb{Z}^n]$), and so $H \cap \mathbb{Z}^n$ is a finite index (free) abelian subgroup of H, proving the claim. □

In the next Lemma, we establish criteria for an element of a virtually abelian group to lie in a chosen subgroup.

Lemma 3.4.4. *Let G be a finitely generated virtually abelian group, with normal subgroup $\mathbb{Z}^n$ of finite index d as usual. Let $H < G$, and choose a set $\{h_1, \ldots, h_c\} \in H$ of coset representatives for $H \cap \mathbb{Z}^n \backslash H$. Then this set can be extended to a set $\{h_1, \ldots, h_c, \ldots, h_d\}$ of coset representatives for $\mathbb{Z}^n \backslash G$, and an element $xh_i \in G$, with $x \in \mathbb{Z}^n$, is in H if and only if*

1. *$1 \leq i \leq c$, and*
2. *$x \in H \cap \mathbb{Z}^n$.*

Proof. Firstly, Lemma 3.4.3 implies that $H \cap \mathbb{Z}^n$ is a finite index subgroup of H, and that $c \leq d$. We have

$$H = \bigsqcup_{i=1}^{c} (H \cap \mathbb{Z}^n)h_i. \tag{3.17}$$

Next, we note that each element $h_i \in \{h_1, \ldots, h_c\}$ represents a unique $\mathbb{Z}^n$-coset: let $h_i \neq h_j$, and suppose that they define the same coset in $G/\mathbb{Z}^n$. Then $h_i h_j^{-1} \in \mathbb{Z}^n$, and since both are elements of H, $h_i h_j^{-1} \in H \cap \mathbb{Z}^n$. But this would imply that they represent the same coset in $H/H \cap \mathbb{Z}^n$, which is a contradiction.

Now choose elements $h_{c+1}, \ldots, h_d$ so that $\{h_1, \ldots, h_d\}$ is a set of coset representatives for $G/\mathbb{Z}^n$, so we have

$$G = \left(\bigsqcup_{i=1}^{c} \mathbb{Z}^n h_i\right) \sqcup \left(\bigsqcup_{i=c+1}^{d} \mathbb{Z}^n h_i\right). \tag{3.18}$$

From this decomposition, it is clear that an element $xh_i \in G$ (with $x \in \mathbb{Z}^n$) is in H if and only if it lies in one of the cosets that intersects H (that is, $h_i \in \{h_1, \ldots, h_c\}$) and the abelian part lies in $H \cap \mathbb{Z}^n$ (that is, $x \in H \cap \mathbb{Z}^n$). □

Proof of Theorem 3.4.2. The strategy of the proof is to use the criteria given in Lemma 3.4.4 to detect those elements of G which are contained in H, pattern by pattern, and show that they form polyhedral sets. We can then use the sets U^π to find minimal representatives for the elements of H.

Fix a generating set S for G, and a weight function ω. We consider the expanded generating set $\widetilde{S}$ of Section **??**, and the corresponding finite set of patterns P as in Definition 3.1.7. Since words with the same pattern represent elements of the same $\mathbb{Z}^n$-coset (see Remark 3.1.13), only words whose pattern represents an element of $\bigsqcup_{i=1}^{c} \mathbb{Z}^n h_i$ can possibly be in H. Call the set of such patterns P_H.

Fix one of these patterns, $\pi \in P_H$, and the resulting vectors A_i^π and integers B_i^π as in section 3.1. Consider a word $w \in W^\pi$. By design, we have an h_i with $1 \leq i \leq c$ so that $\overline{w} \in \mathbb{Z}^n h_i$, so criterion (1) is satisfied. Now by criterion (2), w represents an element of H if and only if $\overline{w} \in (H \cap \mathbb{Z}^n) h_i \subset \mathbb{Z}^n h_i$. Since

$$\overline{w} = \left\{ \begin{pmatrix} A_1^\pi \cdot \vec{w} \\ A_2^\pi \cdot \vec{w} \\ \vdots \\ A_n^\pi \cdot \vec{w} \end{pmatrix} + \begin{pmatrix} B_1^\pi \\ B_2^\pi \\ \vdots \\ B_n^\pi \end{pmatrix} \right\} h_i,$$

$\overline{w} \in H$ if and only if

$$\begin{pmatrix} A_1^\pi \cdot \vec{w} \\ A_2^\pi \cdot \vec{w} \\ \vdots \\ A_n^\pi \cdot \vec{w} \end{pmatrix} + \begin{pmatrix} B_1^\pi \\ B_2^\pi \\ \vdots \\ B_n^\pi \end{pmatrix} \in H \cap \mathbb{Z}^n. \tag{3.19}$$

Now $H \cap \mathbb{Z}^n$ is a (free abelian) subgroup of $\mathbb{Z}^n$. Suppose it has dimension f, and choose a basis $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_f\} \subset \mathbb{Z}^n$. Then w satisfies (3.19) if and only if there exist integers $a_1, \ldots, a_f$ so that

$$\begin{pmatrix} A_1^\pi \cdot \vec{w} \\ A_2^\pi \cdot \vec{w} \\ \vdots \\ A_n^\pi \cdot \vec{w} \end{pmatrix} + \begin{pmatrix} B_1^\pi \\ B_2^\pi \\ \vdots \\ B_n^\pi \end{pmatrix} = a_1 \boldsymbol{b}_1 + \cdots + a_f \boldsymbol{b}_f.$$

In other words,

$$A_i^\pi \cdot \vec{w} + B_i^\pi = e_i \cdot (a_1\boldsymbol{b}_1 + \cdots + a_f\boldsymbol{b}_f) = a_1(e_i \cdot \boldsymbol{b}_1) + \cdots + a_f(e_1 \cdot \boldsymbol{b}_f) \tag{3.20}$$

for each $1 \leq i \leq n$.

We will now express the set of all vectors satisfying (3.20) as a polyhedral set. For each $1 \leq i \leq n$, define the $(m+f)$-dimensional vector

$$C_i^\pi = \begin{pmatrix} | \\ A_i^\pi \\ | \\ -e_i \cdot \boldsymbol{b}_1 \\ \vdots \\ -e_i \cdot \boldsymbol{b}_f \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} | \\ A \\ | \end{matrix}}\right\} m \text{ rows} \\ \left.\vphantom{\begin{matrix} b \\ \vdots \\ b \end{matrix}}\right\} f \text{ rows} \end{array}$$

where the first m entries are the entries of the vector A_i^π, and the last f entries are -1 times the ith component of the basis vectors. Now a vector $\vec{w} \in \mathbb{Z}^m$ satisfies (3.20) for some i precisely when there is a vector $\vec{v} \in \mathbb{Z}^{m+f}$, with entries $v_1, v_2, \ldots, v_{m+f}$, such that $\vec{w} = (v_1, \ldots, v_m)^\top$ and $A_i^\pi \cdot \vec{w} + B_i^\pi = e_i \cdot (v_{m+1}\boldsymbol{b}_1 + \cdots + v_{m+f}\boldsymbol{b}_f)$. We rewrite this last equation as

$$A_i^\pi \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} - \begin{pmatrix} e_i \cdot \boldsymbol{b}_1 \\ e_i \cdot \boldsymbol{b}_2 \\ \vdots \\ e_i \cdot \boldsymbol{b}_f \end{pmatrix} \cdot \begin{pmatrix} v_{m+1} \\ v_{m+2} \\ \vdots \\ v_{m+f} \end{pmatrix} = -B_i^\pi$$

i.e.

$$C_i^\pi \cdot \vec{v} = -B_i^\pi.$$

Let $p_m \colon \mathbb{Z}^{m+f} \to \mathbb{Z}^m$ denote projection onto the first m coordinates. We can then express the set of all $\vec{w}$ that satisfy (3.19) as the following

$$X^\pi = p_m\left(\bigcap_{i=1}^{n} \{\vec{v} \in \mathbb{Z}^{m+f} \mid C_i^\pi \cdot \vec{v} = -B_i^\pi\}\right) \cap \mathbb{N}^m.$$

Note that $\{\vec{v} \in \mathbb{Z}^{m+f} \mid C_i^\pi \cdot \vec{v} = -B_i^\pi\}$ is an elementary set for each i, and therefore the intersection is a (basic) polyhedral set. Therefore X^π is also a polyhedral set. It corresponds to all words in W^π which represent elements of the subgroup H. If we set

$$V_t = \bigcup_{\pi \in P_t} \mathcal{E}^\pi(X^\pi),$$

the result now follows from Theorem 3.4.1. □

Corollary 3.4.5. *Finite unions of subgroups of virtually abelian groups have rational relative growth.*

Proof. First note that the growth series of a finite *disjoint* union of subsets of G is simply the sum of their individual growth series, and for subsets $A \subset B \subseteq G$, the growth series of $B \setminus A$ is the difference of their individual growth series. We will induct on the number of subgroups in the finite union.

For subgroups H_1 and H_2, let $I = H_1 \cap H_2$. Then we can express their union as a disjoint union of three subsets,

$$H_1 \cup H_2 = I \cup (H_1 \setminus I) \cup (H_2 \setminus I).$$

Since H_1, H_2, and I are all subgroups, they have rational relative growth, so each term in the above expression does also, and so $H_1 \cup H_2$ has rational growth series.

Now assume the union of k subgroups has rational growth. Consider a union of $k+1$ subgroups:

$$\bigcup_{i=1}^{k+1} H_i = \bigcup_{i=1}^{k} H_i \cup H_{k+1}.$$

Let $J = \left(\bigcup_{i=1}^{k} H_i\right) \cap H_{k+1}$. Then we have a disjoint union

$$\bigcup_{i=1}^{k+1} H_i = J \cup \left(\bigcup_{i=1}^{k} H_i\right) \setminus J \cup (H_{k+1} \setminus J)$$

and so if J has rational growth then $\bigcup_{i=1}^{k+1} H_i$ has rational growth. But we can write J as the union of k subgroups:

$$J = \bigcup_{i=1}^{k} (H_i \cap H_{k+1})$$

so it has rational growth by the inductive hypothesis. □

Chapter 4

Nilpotent groups

In this chapter we study the cumulative conjugacy growth of some nilpotent groups of class 2. We first recall the essential definitions, using Hall [43] as a basic reference.

Definition 4.0.1. If x_1, x_2 are elements of a group, we define their *commutator* $[x_1, x_2] = x_1 x_2 x_1^{-1} x_2^{-1}$. We can then define the *n-fold commutator* of elements $x_1, x_2, \dots, x_n$ inductively:

$$[x_1, x_2, \dots, x_n] := [[x_1, x_2, \dots, x_{n-1}], x_n].$$

Definition 4.0.2. For subgroups $X_1, X_2, \dots X_n$ of some group, define their n-fold commutator subgroup

$$[X_1, X_2, \dots, X_n] = \langle [x_1, x_2, \dots, x_n] \mid x_i \in X_i \rangle .$$

Definition 4.0.3. Let $G^{(c)} = [\underbrace{G, G, \dots, G}_{c \text{ copies}}]$. Then the *lower central series* of the group G is the subnormal series

$$G = G^{(0)} \rhd G^{(1)} \rhd G^{(2)} \rhd \cdots$$

Remark 4.0.4. *If $\phi \in \mathrm{Aut}(G)$, for some group G, then $\phi([g, h]) = [\phi(g), \phi(h)]$ (for any $g, h \in G$). Thus the derived subgroup of any group G is a characteristic subgroup. By an analogous argument, the ith term in the lower central series is characteristic, for any i.*

Definition 4.0.5. A group is called *nilpotent* if its lower central series terminates. That is, there exists $c \in \mathbb{N}$ such that $G^{(c)}$ is trivial. The minimal such c is called the *nilpotency class* of the group.

The class 1 nilpotent groups are precisely the abelian groups.

Note that each $G^{(i-1)}/G^{(i)}$ is abelian, and furthermore is central in $G/G^{(i)}$. Write $\mathrm{rk}(H)$ for the torsion-free rank of any abelian group H.

Definition 4.0.6. The *Hirsch length* of a class c nilpotent group G is the sum

$$h(G) = \sum_{i=1}^{c} \mathrm{rk}\left(G^{(i-1)}/G^{(i)}\right).$$

Remark 4.0.7. *For $\phi \in \mathrm{Aut}(G)$ we have $\phi([x_1, \ldots, x_r]) = [\phi(x_1), \ldots, \phi(x_r)]$ and therefore the subgroup $G^{(r)}$ is characteristic.*

The following Theorem was proved independently by Bass [3] and Guivarc'h [41], [42].

Theorem 4.0.8. *Let G be nilpotent. Then the (cumulative) growth function of G is polynomial of degree $\sum_{i=1}^{c} i \cdot \mathrm{rk}\left(G^{(i-1)}/G^{(i)}\right)$.*

The following celebrated theorem of Gromov provides the converse to this, cementing the connection between growth and algebraic structure.

Theorem 4.0.9 ([37])**.** *A finitely generated group has polynomial growth if and only if it is virtually nilpotent.*

Definition 4.0.10. The *(higher) Heisenberg groups* are class 2 nilpotent groups, with a parameter $r \in \mathbb{N}_+$, given by the following presentation.

$$H_r := \left\langle a_1, b_1, a_2, b_2, \ldots, a_r, b_r \;\middle|\; \begin{array}{l} [a_i, a_j] = [a_i, b_j] = [b_i, b_j] = 1 \ \forall i \neq j \\ [a_i, b_i] = [a_j, b_j] \ \forall i \neq j \\ [[a_i, b_i], a_j] = [[a_i, b_i], b_j] = 1 \ \forall i, j \end{array} \right\rangle.$$

We will usually write $c = [a_i, b_i]$.

Stoll classified the class 2 nilpotent groups G with $[G, G] \cong \mathbb{Z}$ in terms of the groups H_r. Each such G has a finite-index subgroup isomorphic to some $\mathbb{Z}^m \times H_r$, and this r is called the *Heisenberg rank* of G. Stoll then proved that for $r \geq 2$, G has a generating set for which the standard growth series is rational, and one for which it is transcendental.

By contrast, Duchin and Shapiro [24] have shown that H_1 has rational standard growth with respect to any generating set.

Babenko [2] provided good asymptotic estimates for the *conjugacy* growth of the higher Heisenberg groups.

Theorem 4.0.11 (Theorem 4.1 of [2])**.** *Let ρ be any left-invariant metric on H_r, and write $c_{H_r,\rho}(n)$ for the number of conjugacy classes intersecting the n-ball with respect to ρ. Then*

$$c_{H_1,\rho}(n) = \frac{1}{\zeta(2)} n^2 \log n + o(n^2 \log n)$$

and

$$c_{H_r,\rho}(n) = \frac{\zeta(2r-1)}{\zeta(2r)} n^{2r} + o(n^{2r})$$

for $r \geq 2$, where ζ denotes the Riemann zeta function.

Remark 4.0.12. *Observe that if the conjugacy growth series of H_r were rational, then Proposition 2.3.8 would imply that $\frac{\zeta(2r-1)}{\zeta(2r)}$ was a rational number. To the author's knowledge this is not known, for any integer $r > 1$, but would seem to be unlikely.*

In this chapter we use more elementary methods to recover Babenko's results in the special case where ρ is the word metric arising from some finite generating set, and extend the scope to include all class 2 nilpotent groups with infinite cyclic derived subgroup (Theorem 4.3.3). This comes at a cost: we only recover asymptotics up to the usual equivalence of growth functions, whereas Babenko's original result includes the leading terms.

Babenko also provides a (fairly terse) proof that groups with a finite-index subgroup isomorphic to H_1 also have conjugacy growth equivalent to $n^2 \log n$. We provide a similar proof here with full details (Theorem 4.5.1). It is hoped that the same techniques can be generalised to study the conjugacy growth of other virtually nilpotent groups.

4.1 Centrally amalgamated direct products

This section details Stoll's classification [57] of class 2 nilpotent groups whose derived subgroup is infinite cyclic.

Let $G_1, \ldots, G_r$ be groups with central subgroups $Z_1, \ldots, Z_r$ respectively. Suppose that there exists an abelian group Z and homomorphisms $\varphi_i \colon Z_i \to Z$ for each i, and consider the product homomorphism $\varphi \colon Z_1 \times \cdots \times Z_r \to Z$ defined by

$$\varphi \colon (z_1, \ldots, z_r) \mapsto \varphi_1(z_1) \cdots \varphi_r(z_r).$$

Furthermore, suppose that φ is surjective.

Definition 4.1.1. The group

$$G = \frac{G_1 \times \cdots \times G_r}{\ker \varphi}$$

is called the *centrally amalgamated direct product of* $G_1, \ldots, G_r$ *with respect to* φ. We will write *central product* for brevity.

Let $A_i = G_i/Z_i$ for each i. We have $Z \cong Z_1 \times \cdots \times Z_r / \ker \varphi$. Then let

$$A := G/Z \cong \frac{G_1 \times \cdots \times G_r / \ker \varphi}{Z_1 \times \cdots \times Z_r / \ker \varphi} \cong \frac{G_1 \times \cdots \times G_r}{Z_1 \times \cdots \times Z_r} \cong A_1 \times \cdots \times A_r.$$

Example 4.1.2. *Let $G_1 \cong G_2 \cong \mathbb{Z}^2$ be given by the presentations*

$$G_1 = \langle x_1, y_1 \mid [x_1, y_1] \rangle, \; G_2 = \langle x_2, y_2 \mid [x_2, y_2] \rangle,$$

and let $Z = \langle z \mid z^2 \rangle$. Define homomorphisms from the second direct factor of each G_i to Z: $\varphi_i \colon \langle y_i \rangle \to Z$ given by $\varphi \colon y_i \mapsto z$. Then the centrally amalgamated direct product, G, of G_1 and G_2 with respect to φ is given by the presentation

$$\langle x_1, x_2, z \mid [x_1, x_2], [x_1, z], [x_2, z], z^2 \rangle \cong \mathbb{Z}^2 \times \mathbb{Z}/2\mathbb{Z}.$$

From now on we will deal exclusively with the case where each Z_i is infinite cyclic, generated by an element z_i, and similarly Z is infinite cyclic, generated by an element z. Hence, each φ_i is determined by an integer d_i as follows $\varphi_i \colon z_i \mapsto z^{d_i}$.

Given pairs (G_1, z_1) and (G_2, z_2), we will write $(G_1, z_1) \otimes_d (G_2, z_2)$ for the central product of G_1 and G_2, amalgamated over the subgroups $\langle z_1 \rangle$ and $\langle z_2 \rangle$ with $\varphi_1(z_1) = z$ and $\varphi_1(z_2) = z^d$. If $d = 1$, we simply write $(G_1, z_1) \otimes (G_2, z_2)$.

We now assume that each G_i is 2-step nilpotent with infinite cyclic derived subgroup, and set $Z_i = [G_i, G_i]$. Thus each A_i, and hence A, is abelian. We will use the following classification given by Stoll.

Lemma 4.1.3 (Lemma 7.1 of [57])**.** *Let G be a finitely generated 2-step nilpotent group with $[G, G] \cong \mathbb{Z}$. Then there exists a finitely generated infinite abelian group G_0, and a tuple $D = (\delta_1, \ldots, \delta_{r-1}) \in \mathbb{N}_+$, with $\delta_k | \delta_{k+1}$ for each k, such that*

$$G \cong (\cdots ((((G_0, z) \otimes (H_1, c)) \otimes_{\delta_1} (H_1, c_2)) \otimes_{\delta_2} (H_1, c_3)) \cdots) \otimes_{\delta_{r-1}} (H_1, c_r).$$

Since the abelian group $\Gamma := G_0/\langle z \rangle$ is central, we have the following immediate corollary.

Corollary 4.1.4. *With the assumptions of Lemma 4.1.3, there exists a finitely generated abelian group Γ (possibly finite or trivial) such that*

$$G \cong \Gamma \times \left((\cdots (((H_1, c) \otimes_{\delta_1} (H_1, c_2)) \otimes_{\delta_2} (H_1, c_3)) \cdots) \otimes_{\delta_{r-1}} (H_1, c_r) \right).$$

Definition 4.1.5. We will write

$$H_D = (\cdots (((H_1, c) \otimes_{\delta_1} (H_1, c_2)) \otimes_{\delta_2} (H_1, c_3)) \cdots) \otimes_{\delta_{r-1}} (H_1, c_r).$$

Thus, given an abelian group Γ, and an $(r-1)$-tuple D of positive integers (for $r \geq 1$) where each entry divides the next, we can define a group $\Gamma \times H_D$. All finitely generated nilpotent groups with infinite cyclic derived subgroup are of this form. If Γ is trivial and each $\delta_k = 1$, the corresponding Stoll group is the rth (higher) Heisenberg group, with H_1 corresponding to the case where $r = 1$ (i.e. D is empty). That is,

$$H_r = \underbrace{(H_1, c) \otimes (H_1, c_2) \otimes \cdots \otimes (H_1, c_r)}_{r \text{ copies of } H_1}.$$

We will need the following observation.

Lemma 4.1.6. *For any group H_D as above, the centre $Z(H_D) = \langle c \rangle$ has quadratic relative (cumulative) growth.*

Proof. We work with the generating set $\{a_1, b_1, \dots, a_r, b_r\}$, and write $B(n)$ for the ball of radius n in the Cayley graph of H_D. Note that for any i we have $[a_i^m, b_i^n] = c^{\delta_{i-1}mn}$. Write $C(n) = \{c^l \mid |l| \leq n\}$. Let $c^k \in C(n^2)$, and suppose $k > 0$. So we can write $k = n^2 - \alpha n - \beta$ for some $0 \leq \alpha, \beta < n$. We then have

$$a_1^{n-\alpha} b_1^{n-\beta} a_1^{-1} b_1^{\beta} a_1^{-(n-\alpha-1)} b_1^{-n} = a^{n-\alpha} b^n a^{-(n-\alpha)} b^{-n} c^{-\beta} = c^{n^2-\alpha n-\beta} = c^k$$

and thus $|c^k| < (n-\alpha) + (n-\beta) + 1 + \beta + (n-\alpha-1) + n = 4n - 2\alpha$. An exactly symmetrical argument deals with the case of k negative. Therefore any element of $C(n^2)$ has length at most $4n$, i.e. $C(n^2) \subset Z(H_D) \cap B(4n)$.

On the other hand, the highest possible power of c, written as a word over $4n$ generators, is $[a_r^n, b_r^n] = c^{\delta_{r-1}n^2}$. Thus $Z(H_D) \cap B(4n) \subset C(\delta_{r-1}n^2)$. So together we have

$$C\left(\frac{n^2}{16}\right) \subset Z(H_D) \cap B(n) \subset C\left(\delta_{r-1}\frac{n^2}{16}\right)$$

and so

$$2\frac{n^2}{16} + 1 \leq |Z(H_D) \cap B(n)| \leq 2\delta_{r-1}\frac{n_2}{16} + 1$$

i.e. $|Z(H_D) \cap B(n)| \sim n^2$ as required. □

4.2 Greatest Common Divisors

To count conjugacy classes we will need various facts about greatest common divisors of tuples of integers, starting with the following lemma of Fernández and Fernández.

Lemma 4.2.1 ([29])**.** *For $n \geq 1$, let $X_1^{(n)}, X_2^{(n)}, \dots$ be a sequence of independent random variables, uniformly distributed in $\{1, 2, \dots, n\}$. Then the expected value of the greatest common divisor of the first s of these random variables behaves as* *follows.*

$$\mathbb{E}\left(\gcd\left(X_1^{(n)}, X_2^{(n)}, \ldots, X_s^{(n)}\right)\right) = \begin{cases} \frac{1}{\zeta(2)} \log n + C + \mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right) & s = 2 \\ \frac{\zeta(s-1)}{\zeta(s)} + \mathcal{O}\left(\frac{\log n}{n}\right) & s \geq 3 \end{cases}$$

where $C \geq 0$ is some constant.

For our purposes we will phrase this in terms of the sum of the greatest common divisors of tuples of integers whose values are at most n.

Definition 4.2.2. We define two different n-balls in $\mathbb{Z}^s$.

1. Let $B_{\square}^{(s)}(n) = \{(x_1, \ldots, x_s) \in \mathbb{Z}^s \mid |x_i| \leq n \text{ for each } 1 \leq i \leq s\}$. That is, the n-ball in $\mathbb{Z}^s$ with respect to the 'cubical' generating set $\{(\epsilon_1, \ldots, \epsilon_s) \mid \epsilon_i \in \{0, 1\}\}$.

2. Let $B_{l_1}^{(s)}(n) = \{(x_1, \ldots, x_s) \in \mathbb{Z}^s \mid \sum |x_i| \leq n\}$. That is, the n-ball in $\mathbb{Z}^s$ with respect to the generating set consisting of standard basis vectors.

We will omit the superscript s when it is clear which dimension we are working with.

Then Lemma 4.2.1 can be reinterpreted as follows.

Corollary 4.2.3.

$$\sum_{(x,y) \in B_{\square}^{(2)}(n)} \gcd(x, y) = \frac{R_2}{\zeta(2)} n^2 \log n + O(n^2)$$

where $R_2 \in \mathbb{Q}$. And

$$\sum_{(x_1, \ldots, x_s) \in B_{\square}^{(s)}} \gcd(x_1, \ldots, x_s) = R_s \frac{\zeta(s-1)}{\zeta(s)} n^s + \mathcal{O}(n^{s-1} \log n)$$

where $R_s \in \mathbb{Q}$ depends on the dimension s.

Proof. The sum of the values of a function over some fixed finite domain is equal to the expected value of the function over the domain, multiplied by the cardinality of the domain. The cumulative growth function of $\mathbb{Z}^s$ is equivalent to Cn^s (Proposition 2.2.7), where C depends on the choice of generating set, but is always rational since otherwise Proposition 2.3.8 would imply that the standard growth series was irrational, contradicting Benson's main result (Theorem 3.2.8).

□

We will also need the following consequence of Lemma 4.2.1.

Corollary 4.2.4. *Let $(a,b) \in \mathbb{Z}^2$ be fixed. Then*

$$\sum_{(x,y)\in B_\square^{(s)}(n)} \gcd(x-a, y-b) \sim n^2 \log n.$$

Proof. Assume $a, b \geq 0$. The other cases will follow by symmetry, since $\gcd(-a,b) = \gcd(a,-b) = \gcd(-a,-b) = \gcd(a,b)$. We have

$$\sum_{(x,y)\in B_\square^{(s)}(n)} \gcd(x-a,y-b) = \sum_{(x,y)\in B_\square^{(s)}(n)} \gcd(x,y) + \sum_{(x,y)\in A} \gcd(x,y) - \sum_{(x,y)\in B} \gcd(x,y) \tag{4.1}$$

where $A = \{(x,y) \in \mathbb{Z}^2 \mid |x-a| \leq n,\ |y-b| \leq n\} \setminus B_\square(n)$ and $B = B_\square(n) \setminus \{(x,y) \in \mathbb{Z}^2 \mid |x-a| \leq n,\ |y-b| \leq n\}$.

We have $|A| < (2n+1)(a+b)$ and $|B| < (2n+1)(a+b)$. Furthermore, for $(x,y) \in A \cup B$, $\gcd(x,y) \leq \max(n+a, n+b) = n + \max(a,b)$. Therefore

$$\sum_{(x,y)\in A} \gcd(x,y) < (n + \max(a,b))(2n+1)(a+b) \preccurlyeq n^2$$

and similarly for B. So since $\sum_{(x,y)\in B_\square(n)} \gcd(x,y) \sim n^2 \log n$, equation (4.1) gives the result. □

We now define a weighted form of the greatest common divisor tailored to our needs.

Definition 4.2.5. Let $D = (\delta_1, \delta_2, \ldots, \delta_{r-1}) \in \mathbb{N}_+^{r-1}$, and $\alpha = (i_1, j_1, i_2, j_2, \ldots, i_r, j_r) \in \mathbb{Z}^{2r}$. Then define

$$g_D(\alpha) = \gcd(i_1, j_1, \delta_1 i_2, \delta_1 j_2, \delta_2 i_3, \delta_2 j_3, \ldots, \delta_{r-1} i_r, \delta_{r-1} j_r).$$

Lemma 4.2.6. *Let $\mathcal{H} \subset \mathbb{Z}^{2r}$ be the union of hyperplanes that contain the origin,* *and are perpendicular to an axis, i.e.*

$$\mathcal{H} = \{z \in \mathbb{Z}^{2r} \mid \exists i \ s.t. \ z_i = 0\}.$$

Fix some $D = (\delta_1, \delta_2, \ldots, \delta_{r-1}) \in \mathbb{N}_+^{r-1}$ *as above. Then*

$$\sum_{\alpha \in \mathcal{H} \cap B_\square(n)} g_D(\alpha) \preccurlyeq n^{2r}.$$

Proof. Fix a coordinate to be zero (there are $2r$ choices). There are then $(2n+1)^{2r-1}$ ways to complete the entries of α. Thus $|\mathcal{H} \cap B_\square(n)| = 2r(2n+1)^{2r-1} = \mathcal{O}(n^{2r-1})$. For a general element $\alpha \in B_\square(n)$, we have $g_D(\alpha) \leq \delta_{\max} n$ where $\delta_{\max} = \max\{\delta_i \mid 1 \leq i \leq r-1\}$. Thus we have

$$\sum_{\alpha \in \mathcal{H} \cap B_\square(n)} g_D(\alpha) \leq 2r(2n+1)^{2r-1} \cdot \delta_{\max} n$$

and the result follows. □

Theorem 4.2.7. *Let* $r \geq 1$*. Fix an* $(r-1)$*-tuple* D *as above. Let* $\alpha \in \mathbb{Z}^{2r}$*. Then if* $r = 1$ *we have*

$$\sum_{\alpha \in B_{l_1}^{(2)}(n)} g_D(\alpha) \sim n^2 \log n,$$

and if $r > 1$ *we have*

$$\sum_{\alpha \in B_{l_1}^{(2r)}(n)} g_D(\alpha) \sim n^{2r}.$$

Proof. By symmetry, we only need consider α with non-negative coordinates (up to a constant factor of 2^r). Additionally, we will assume that the coordinates of α are all positive, since the other case is covered by Lemma 4.2.6.

Let $g_1(\alpha) = \gcd(i_1, j_1, i_2, j_2, \ldots, i_r, j_r)$, i.e. $g_1 = g_D$ where all the entries of D are equal to 1. We show that it suffices to prove the proposition for this special case.

Consider general D. Observe first that for any $x, y, z \in \mathbb{Z}$ we have

$$\gcd(x, y) \leq \gcd(x, yz) \leq \gcd(xz, yz) = z \gcd(x, y).$$

Since the greatest common divisor is an associative binary operation, and each

$\delta_k|\delta_{k+1}$, this gives the following bounds for $g_D(\alpha)$, for all $\alpha \in \mathbb{Z}^{2r}$:

$$g_1(\alpha) \leq g_D(\alpha) \leq \delta_{r-1} g_1(\alpha).$$

This in turn implies that

$$\sum_{\alpha \in B_{l_1}(n)} g_1(\alpha) \leq \sum_{\alpha \in B_{l_1}(n)} g_D(\alpha) \leq \delta_{r-1} \sum_{\alpha \in B_{l_1}(n)} g_1(\alpha)$$

and so

$$\sum_{\alpha \in B_{l_1}(n)} g_D(\alpha) \sim \sum_{\alpha \in B_{l_1}(n)} g_1(\alpha).$$

From now on we only consider $g_1(\alpha)$.

Next we observe that counting in the l_1 metric is equivalent to counting in the cubular metric. Since $B_\square\left(\frac{n}{2r}\right) \subset B_{l_1}(n) \subset B_\square(n)$, we have

$$\sum_{\alpha \in B_\square(\frac{n}{2r})} g_1(\alpha) < \sum_{\alpha \in B_{l_1}(n)} g_1(\alpha) < \sum_{\alpha \in B_\square(n)} g_1(\alpha),$$

and so

$$\sum_{\alpha \in B_{l_1}(n)} g_1(\alpha) \sim \sum_{\alpha \in B_\square(n)} g_1(\alpha).$$

Corollary 4.2.3 now gives the result.

$\square$

4.3 Conjugacy Growth

We now return to the group H_D, with generators a_i, b_i and commutator c as in Definition 4.1.5. The next proposition describes the structure of conjugacy classes in H_D.

Proposition 4.3.1. *Let $\alpha = a_1^{i_1} b_1^{j_1} a_2^{i_2} b_2^{j_2} \cdots a_r^{i_r} b_r^{j_r} \in \mathrm{Ab}(H_D) \cong \mathbb{Z}^{2r}$, and consider the element $\alpha c^k \in H_D$ for some $k \in \mathbb{Z}$. Then the conjugacy class represented by*

$\alpha c^k \in H_D$ *is either a singleton set, or a coset of a cyclic subgroup:*

$$[\alpha c^k] = \begin{cases} \{c^k\} & \alpha = 0 \\ \alpha c^k \langle c^{g_D(\alpha)} \rangle & \alpha \neq 0. \end{cases}$$

Proof. First, c^k is central, so $[c^k] = \{c^k\}$. Now consider non-identity α. Note that by the construction of H_D, we have

$$[a_t, b_t] = c_t = \begin{cases} c & t = 1 \\ c^{\delta_{t-1}} & t > 1 \end{cases}.$$

Thus, for $1 \leq t \leq r$, we have

$$\begin{aligned} a_t \alpha c^k a_t^{-1} &= a_t a_1^{i_1} b_1^{j_1} \cdots a_r^{i_r} b_r^{j_r} c^k a_t^{-1} \\ &= a_1^{i_1} b_1^{j_1} \cdots a_t^{i_t+1} b_t^{j_t} a_t^{-1} \cdots a_r^{i_r} b_r^{j_r} c^k \\ &= a_1^{i_1} b_1^{j_1} \cdots a_t^{i_t} b_t^{j_t} \cdots a_r^{i_r} b_r^{j_r} c^{k + j_t \delta_{t-1}}. \end{aligned}$$

Similarly, we have

$$b_t \alpha c^k b_t^{-1} = \alpha c^{k - i_t \delta_{t-1}}.$$

Thus we have

$$\begin{aligned} [\alpha c^k] &= \{\alpha c^{k + \sum_{t=1}^r \delta_{t-1}(l_{t1} j_t - l_{t2} i_t)} \mid l_{t1}, l_{t2} \in \mathbb{Z}\} \\ &= \alpha c^k \left\langle c^{\gcd(j_1, -i_1, \delta_1 j_2, -\delta_1 i_2, \ldots, \delta_{r-1} j_r, -\delta_{r-1} i_r)} \right\rangle \\ &= \alpha c^k \langle c^{g_D(\alpha)} \rangle. \end{aligned}$$

□

Now we estimate the length of the conjugacy classes with respect to the generating set $\{a_1^{\pm 1}, b_1^{\pm 1}, \ldots, a_r^{\pm 1}, b_r^{\pm 1}\}$.

Proposition 4.3.2. *Let $G = H_D$, and let $\alpha = a_1^{i_1} b_1^{j_1} a_2^{i_2} b_2^{j_2} \cdots a_r^{i_r} b_r^{j_r}$ be a non-trivial element of the abelianisation.*

Then there exists $K \geq 0$ such that $|\alpha| \leq |[\alpha c^k]| \leq |\alpha| + K$ for all $k \in \mathbb{Z}$. This K depends only on the tuple D, so is a constant for a given group.

Proof. Without loss of generality, we assume that each coordinate of α is non-negative. The full result will follow by isometries of (the Cayley graph of) G which send some subset of the generators to their inverses. First, we claim that any element αc^k has length at least $|\alpha|$, so $|\alpha| \leq |[\alpha c^k]|$. To see this, consider any word over the generators $a_1, b_1, \ldots, a_r, b_r$ that represents αc^k. We can put this into normal form by collecting the powers of generators into the given order, at the cost of powers of c, using the identities $[a_t, b_t] = c^{\delta_{t-1}}$ for each t. Note that the exponent sum of each generator can never increase. So any word representing αc^k has at least i_1 instances of a_1, and so on.

From the structure of conjugacy classes given in Proposition 4.3.1, each conjugacy class $[\alpha c^k]$ has a representative of the form αc^{-l} where $0 \leq l < g_D(\alpha)$. We will show that there is a global K such that all such elements have length at most $|\alpha| + K$, which proves the proposition.

By definition of the greatest common divisor, we have

$$g_D(\alpha) \leq \min\{\delta_k i_k, \delta_k j_k \mid 1 \leq k \leq r, i_k \neq 0, j_k \neq 0\}.$$

Choose some i_k or j_k which is non-zero. We will suppose it is an i_k but the argument works analogously in the second case.

Suppose $j_k \neq 0$. For any $1 \leq I_k \leq i_k$ we have

$$a_k^{i_k - I_k} b_k a_k^{I_k} b_k^{j_k - 1} =_G a_k^{i_k} [a_k^{-I_k}, b_k] b_k^{j_k} =_G a_k^{i_k} b_k^{j_k} c^{-I_k \delta_k} \tag{4.2}$$

and so the element represented by $a_k^{i_k} b_k^{j_k} c^{-I_k \delta_k}$ has length $i_k + j_k$, and consequently the element $\alpha c^{-I_k \delta_k}$ has length $|\alpha|$, for all $1 \leq I_k \leq i_k$. Any element of the form αc^{-l}, where $0 \leq l < g_D(\alpha)$, can be expressed as $\alpha c^{-I_k \delta_k} c^{\epsilon}$ for some $1 \leq I_k \leq i_k$ and $0 \leq \epsilon < \delta_k$. We have $c^{\epsilon} = [a_1^{\epsilon}, b_1]$, so $|c^{\epsilon}| \leq 2\epsilon + 2$, and thus our element has length at most $|\alpha| + 2\epsilon + 2$.

Now suppose $j_k = 0$. To use the identity (4.2) we need to introduce an extra copy of b_k and b_k^{-1}, which adds 2 to the length of the word. Therefore in general we have the bound $|\alpha c^{-l}| \leq |\alpha| + 2\epsilon + 4$. Since $\epsilon \leq \delta_k \leq \delta_{r-1}$, setting $K = 2\delta_{r-1} + 4$ is enough to prove the proposition. $\square$

We can now calculate asymptotics for the conjugacy growth of any group of the form $\Gamma \times H_D$.

Theorem 4.3.3. *Let G be a class 2 nilpotent group, with infinite cyclic derived subgroup, and let its Heisenberg rank be $r \geq 1$. Then there exists $s \in \mathbb{N}$ such that*

$$c_G(n) \sim \begin{cases} n^{2+s} \log n & r = 1 \\ n^{2r+s} & r \geq 2. \end{cases}$$

Corollary 4.3.4. *If G is a class 2 nilpotent group with infinite cyclic derived subgroup, with Heisenberg rank equal to 1, then the conjugacy growth series of G is transcendental (with respect to any finite generating set).*

Proof. Recall part of the ratio test for convergence of an infinite series $\sum_{n\geq 0} a_n$: if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ then the series $\sum_{n\geq 0} a_n$ converges. From Theorem 4.3.3 we have $c_G(n) \sim n^{2+s}\log n$, and therefore $\frac{c_G(n+1)}{c_G(n)} \to 1$ as $n \to \infty$. So we have

$$\left|\frac{c_G(n+1)z^{n+1}}{c_G(n)z^n}\right| \xrightarrow{n\to\infty} |z|.$$

Thus the series $\sum c_G(n)z^n$ converges as long as $|z| < 1$. So by Fatou's Theorem 2.3.9, it is either rational or transcendental.

Suppose it is rational. Now since $c_G(n)$ is bounded above by a polynomial, and is non-decreasing (since it is a cumulative growth function), Corollary 2.3.7 implies that, in fact, there is some $d \in \mathbb{N}$ for which $c_G(n) \sim n^d$. Since this is not the case, the series cannot be rational and therefore must be transcendental. Note that the asymptotics do not depend on the choice of generating set (by Proposition 2.4.1) and therefore transcendence holds for all generating sets. $\square$

Proof of Theorem 4.3.3. First, suppose $G = H_D$ for some tuple $D = (\delta_1, \delta_2, \ldots, \delta_{r-1})$ where $\delta_k \mid \delta_{k+1}$ for each k. For a fixed non-identity $\alpha \in \mathrm{Ab}(H_D)$, Proposition 4.3.1 implies that there are exactly $g_D(\alpha)$ conjugacy classes in the coset $\alpha\langle c\rangle$, and Proposition 4.3.2 implies that they have length in the range $[|\alpha|, |\alpha| + K]$ for some $K \geq 0$

depending only on D. The conjugacy classes corresponding to $\alpha = 1 \in \mathrm{Ab}(H_D)$ are simply the elements of the centre. Thus the conjugacy growth function is bounded as follows:

$$\beta_{Z(H_D)}(n) + \sum_{\alpha \in B_{l_1}(n-K)} g_D(\alpha) \leq c(n) \leq \beta_{Z(H_D)}(n) + \sum_{\alpha \in B_{l_1}(n)} g_D(n).$$

We have $\beta_{Z(H_D)}(n) \sim n^2$ for any H_D, so from Theorem 4.2.7 we have

$$c_{H_D}(n) \sim \begin{cases} n^2 \log n & r = 1 \\ n^{2r} & r \geq 2 \end{cases}$$

Now a general step 2 nilpotent group G with infinite cyclic derived subgroup has the form $\Gamma \times H_D$ for an abelian group Γ and some D. Any abelian group has conjugacy growth function equivalent to n^s for some $s \in \mathbb{N}$ and so by Lemma 2.4.3, we have $c_G(n) \sim n^s \cdot c_{H_D}(n)$, which proves the Theorem. □

4.4 Automorphisms of the Heisenberg group

In order to understand the conjugacy growth of finite extensions of H_1, we need to understand its automorphisms.

Lemma 4.4.1. *Let G be a group with a characteristic subgroup N. Then the natural homomorphism $p\colon G \to G/N$ induces a homomorphism $\phi\colon \mathrm{Aut}(G) \to \mathrm{Aut}(G/N)$.*

Proof. Let $\theta \in \mathrm{Aut}G$. We define $\phi_\theta \in \mathrm{Aut}(G/N)$ as follows. For some element $x \in G/N$, choose a lift $\gamma \in G$. Then define $\phi_\theta(x) = p(\theta(\gamma))$. Suppose γ' is a different lift of x. So $\gamma^{-1}\gamma' \in \mathrm{Ker}(p) = N$, and so $\theta(\gamma^{-1}\gamma') \in N$ since θ is characteristic. Thus $p(\theta(\gamma^{-1}\gamma')) = 1$ and we have $p(\theta(\gamma)) = p(\theta(\gamma))p(\theta(\gamma^{-1}\gamma')) = p(\theta(\gamma'))$. So ϕ_θ is well-defined.

We also have

$$\phi_{\theta_1} \circ \phi_{\theta_2}(x) = \phi_{\theta_1}(p(\theta_2(\gamma))) = p(\theta_1 \circ \theta_2(\gamma)) = \phi_{\theta_1 \circ \theta_2}(x),$$

where the second equality holds because $\theta_2(\gamma)$ is a lift of $p(\theta_2(\gamma))$. Hence $\phi_{\theta_1} \circ \phi_{\theta_2} = \phi_{\theta_1 \circ \theta_2}$ so ϕ is a homomorphism. □

The proof of the following theorem follows that given by Osipov [50], with thanks to Alan Logan for the simplified argument for exactness at $\mathrm{GL}_2(\mathbb{Z})$.

Theorem 4.4.2. *Let H_1 denote the Heisenberg group as usual. Then its automorphism group fits into the following short exact sequence:*

$$1 \to \mathbb{Z}^2 \xrightarrow{\iota} \mathrm{Aut}(H_1) \xrightarrow{\phi} \mathrm{GL}_2(\mathbb{Z}) \to 1$$

where $\mathbb{Z}^2 \cong \mathrm{Inn}(H_1)$, ι is the inclusion map, and ϕ is the homomorphism induced by the abelianisation map $H_1 \to \mathbb{Z}^2$, as in Lemma 4.4.1.

Proof. First, note that we have $\mathrm{Inn}(H_1) \cong H_1/Z(H_1) \cong \langle a, b\rangle \cong \mathbb{Z}^2$. Now we show that the sequence is exact. An inner automorphism will always induce the identity automorphism on the abelianisation, so ϕ maps any inner automorphism to the identity in $\mathrm{GL}_2(\mathbb{Z})$. So $\mathrm{Im}(\iota) \subseteq \mathrm{Ker}(\phi)$. Conversely, let $f \in \mathrm{Ker}(\phi)$. So there exist $\alpha, \beta \in \mathbb{Z}$ such that $f(a) = ac^\alpha$ and $f(b) = bc^\beta$. Consider the element $a^\beta b^{-\alpha}$. We have

$$a^\beta b^{-\alpha} \cdot a \cdot b^\alpha a^{-\beta} = ac^\alpha$$

$$a^\beta b^{-\alpha} \cdot b \cdot b^\alpha a^{-\beta} = bc^\beta.$$

So the inner automorphism defined by conjugation by $a^\beta b^{-\alpha}$ coincides with f on the generators of H_1, so they are the same automorphism. Thus $\mathrm{Ker}(\phi) \subseteq \mathrm{Im}(\iota) = \mathrm{Inn}(H_1)$, and so the sequence is exact at $\mathrm{Aut}(H_1)$.

For exactness at $\mathrm{GL}_2(\mathbb{Z})$, we use the standard fact that $\mathrm{GL}_2(\mathbb{Z})$ is generated by the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Define maps $H_1 \to H_1$, via their values on the generators as follows:

$$\theta_1 \colon \begin{cases} a \mapsto ab \\ b \mapsto b \end{cases}, \quad \theta_2 \colon \begin{cases} a \mapsto b \\ b \mapsto a. \end{cases}$$

Since $[ab, b] = [ab^{-1}, b] = [a, b]$, and $[b, a] = [a, b]^{-1}$, we have $[[ab, b], a] = [[ab, b], b] = 1$, $[[ab^{-1}, b], a] = [[ab^{-1}, b], b] = 1$, and $[[b, a], a] = [[b, a], b] = 1$. Therefore θ_1 and θ_2

are automorphisms with inverses as follows:

$$\theta_1^{-1}\colon \begin{cases} a \mapsto ab^{-1} \\ b \mapsto b \end{cases}, \quad \theta_2^{-1} = \theta_2.$$

It is clear that θ_1 and θ_2 are sent to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ respectively by ϕ, and hence ϕ is surjective. $\square$

We now describe a general automorphism of H_1 more explicitly.

Proposition 4.4.3. *Let $\theta \in \mathrm{Aut}(H_1)$. Then there exist integers $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ (depending only on θ) such that for any $a^i b^j c^k \in H_1$ we have*

$$\theta(a^i b^j c^k) = a^{\alpha i + \beta j} b^{\gamma i + \delta j} c^{kd - \alpha\gamma\frac{i(i-1)}{2} - \beta\delta\frac{j(j-1)}{2} - ij\beta\gamma + \epsilon(\gamma i + \delta j) - \zeta(\alpha i + \beta j)},$$

where $d = \alpha\delta - \beta\gamma$.

Proof. By Theorem 4.4.2, we have $\mathrm{Aut}(H_1)/\mathrm{Inn}(H_1) \cong \mathrm{GL}_2(\mathbb{Z})$. We can choose, as a right transversal, the set of automorphisms $\{\theta_D \mid D \in \mathrm{GL}_2(\mathbb{Z})\}$ corresponding to matrices $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $d = \det D = \pm 1$, where $\theta_D\colon a \mapsto a^\alpha b^\gamma$ and $\theta_D\colon b \mapsto a^\beta b^\delta$. So we can express the automorphism group as follows:

$$\mathrm{Aut}(H_1) = \bigcup_{D \in \mathrm{GL}_2(\mathbb{Z})} (\mathrm{Inn}(H_1))\, \theta_D.$$

So any element of $\mathrm{Aut}(H_1)$ has the form $c_{(\epsilon,\zeta)} \circ \theta_D$ where $c_{(\epsilon,\zeta)}$ is the inner automorphism defined by conjugation by $a^\epsilon b^\zeta$ and $D \in \mathrm{GL}_2(\mathbb{Z})$ as above. We calculate the image of an element $a^i b^j c^k \in H_1$ under such an automorphism. First note that

$$\theta_D(c) = \theta_D([a,b]) = [a^\alpha b^\gamma, a^\beta b^\delta] = a^\alpha b^\gamma a^\beta b^\delta b^{-\gamma} a^{-\alpha} b^{-\delta} a^{-\beta} = c^{\alpha\delta - \gamma\beta} = c^d.$$

So

$$\begin{aligned}\theta_D(a^ib^jc^k) &= (a^\alpha b^\gamma)^i(a^\beta b^\delta)^jc^{kd} \\ &= a^{\alpha i}b^{\gamma i}c^{-\alpha\gamma-2\alpha\gamma-\cdots-(i-1)\alpha\gamma}a^{\beta j}b^{\delta j}c^{-\beta\delta-2\beta\delta-\cdots-(j-1)\beta\delta}c^{kd} \\ &= a^{\alpha i}b^{\gamma i}a^{\beta j}b^{\delta j}c^{kd-\alpha\gamma\frac{i(i-1)}{2}-\beta\delta\frac{j(j-1)}{2}} \\ &= a^{\alpha i+\beta j}b^{\gamma i+\delta j}c^{kd-\alpha\gamma\frac{i(i-1)}{2}-\beta\delta\frac{j(j-1)}{2}-ij\beta\gamma}.\end{aligned}$$

We also note the following basic conjugacy calculation:

$$a^\epsilon b^\zeta \cdot a^Ib^J \cdot b^{-\zeta}a^{-\epsilon} = a^Ib^Jc^{\epsilon J-\zeta I}.$$

Combining these we now have

$$\begin{aligned}c_{(\epsilon,\zeta)}\circ\theta_D(a^ib^jc^k) &= c_{(\epsilon,\zeta)}\left(a^{\alpha i+\beta j}b^{\gamma i+\delta j}c^{kd-\alpha\gamma\frac{i(i-1)}{2}-\beta\delta\frac{j(j-1)}{2}-ij\beta\gamma}\right) \\ &= a^{\alpha i+\beta j}b^{\gamma i+\delta j}c^{kd-\alpha\gamma\frac{i(i-1)}{2}-\beta\delta\frac{j(j-1)}{2}-ij\beta\gamma+\epsilon(\gamma i+\delta j)-\zeta(\alpha i+\beta j)}.\end{aligned}$$

□

Remark 4.4.4. *Under the map ϕ in the proof of Theorem 4.4.2, θ_D induces the linear map $D \in \mathrm{GL}_2(\mathbb{Z})$.*

4.5 Virtually Heisenberg

This section is devoted to the following, which forms part of Theorem 5.1 of [2].

Theorem 4.5.1. *If G has a finite index subgroup isomorphic to the Heisenberg group H_1, then G has cumulative conjugacy growth function equivalent to $n^2\log n$.*

The proof here broadly follows the argument given in [2], but provides full details. We start with the following observation.

Lemma 4.5.2. *If a group G has a finite-index subgroup isomorphic to H_1, it contains a finite index* characteristic *subgroup isomorphic to H_1.*

Proof. Lemma 2.1.1 gives a finite-index characteristic subgroup of G that is contained in the first H_1 (and so is a characteristic subgroup of H_1). The lemma will follow if *every* finite-index subgroup of H_1 is isomorphic to H_1.

So suppose Γ is such a subgroup. Since the torsion-free ranks of the quotients in the lower central series are a quasi-isometry invariant (Theorem 6.3.2), those of Γ are equal to those of H_1. So Γ is a torsion-free, class 2, nilpotent group, $[\Gamma, \Gamma]$ of rank 1, and $\Gamma/[\Gamma, \Gamma]$ of rank 2. So Stoll's classification above implies that Γ must be isomorphic to H_1. □

Proof of Theorem 4.5.1. Firstly, by Lemma 2.4.5, we have $c_G(n) \succcurlyeq c_{H_1}(n) \sim n^2 \log n$. So only an upper bound remains to be shown.

We will consider each H_1-coset separately, and only consider conjugation by elements of the H_1 subgroup. This is sufficient for an upper bound on the conjugacy growth, since further conjugation can only reduce the number of conjugacy classes.

We have the following short exact sequence (where H_1 is normal in G by Lemma 4.5.2):

$$1 \to H_1 \to G \to \Delta \to 1 \tag{4.3}$$

where Δ is some finite group.

Fix a transversal T so that each coset in G/H_1 has the form tH_1 for some $t \in T$. Consider conjugating an element $th \in tH_1$ by some $x \in H_1$:

$$xthx^{-1} = tt^{-1}xthx^{-1} = t\left(\theta_t(x)hx^{-1}\right)$$

where $\theta_t \colon x \mapsto t^{-1}xt$ is an automorphism of H_1 (since H_1 is normal in G). Thus for a fixed coset tH_1, conjugation by the elements of H_1 can be understood as twisted conjugation within H_1.

By Proposition 4.4.3, θ_t can be expressed as follows.

$$\theta_t(a^i b^j c^k) = a^{\alpha i+\beta j} b^{\gamma i+\delta j} c^{kd-\alpha\gamma\frac{i(i-1)}{2}-\beta\delta\frac{j(j-1)}{2}-ij\beta\gamma+\epsilon(\gamma i+\delta j)-\zeta(\alpha i+\beta j)}.$$

So if we have $x = a^i b^j c^k$ and $h = a^l b^m c^n$ (and so $x^{-1} = c^{-k}b^{-j}a^{-i} = a^{-i}b^{-j}c^{-k-ij}$), then

$$\theta_t(x)hx^{-1} = a^{l+(\alpha-1)i+\beta j} b^{m+\gamma i+(\delta-1)j} c^{n+f(i,j)} \tag{4.4}$$

where

$$f(i,j) = k(d-1) - \alpha\gamma\frac{i(i-1)}{2} - \beta\delta\frac{j(j-1)}{2} + (\epsilon + i - l)(\gamma i + \delta j) - ij\beta\gamma - ij + im - \zeta(\alpha i + \beta j).$$

For a fixed $h \in H_1$, equation (4.4) defines a map $\mathbb{Z}^3 \to \mathbb{Z}^3$ via $x \mapsto \theta_t(x)hx^{-1}$. Switching to vector notation, with $h = a^l b^m c^n = (l,m,n)^T$ and $x = a^i b^j c^k = (i,j,k)^T$, this map becomes

$$\begin{pmatrix} i \\ j \\ k \end{pmatrix} \mapsto \begin{pmatrix} (\alpha-1)i + \beta j \\ \gamma i + (\delta - 1)j \\ f(i,j) \end{pmatrix} + \begin{pmatrix} l \\ m \\ n \end{pmatrix}.$$

Projecting onto the first two coordinates gives an affine linear map:

$$\begin{pmatrix} i \\ j \end{pmatrix} \mapsto \begin{pmatrix} \alpha - 1 & \beta \\ \gamma & \delta - 1 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} l \\ m \end{pmatrix}.$$

Recall from the previous section that θ_t induces a linear map $D \in \mathrm{Aut}(H_1/Z(H_1))$, given by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Write the determinant of $D - I_2$ as $d_1 = (\alpha-1)(\delta-1) - \beta\delta = d - \alpha - \delta + 1$, where $d = \det D$. So

$$\alpha + \delta = d - d_1 + 1 \tag{4.5}$$

We consider the possible eigenvalues of the linear map D . If $\det(D - \lambda I) = \lambda^2 - \lambda(\alpha+\delta) + d = 0$ then

$$\lambda = \frac{\alpha + \delta \pm \sqrt{(\alpha+\delta)^2 - 4d}}{2} = \frac{d - d_1 + 1 \pm \sqrt{(d - d_1 + 1)^2 - 4d}}{2}. \tag{4.6}$$

Furthermore, since Δ is a finite group, D must have finite order, so λ is a root of unity. We consider cases depending on the value of the determinants d and d_1. Since $D \in \mathrm{GL}_2(\mathbb{Z})$, $d \in \{1, -1\}$.

Case $d = 1$, $d_1 = 0$

From (4.6), we have $\lambda = 1$ so D has a fixed non-zero point. Since $d = 1$, D is not a reflection, but it has finite order so must be the identity. And so $\theta_t \in \mathrm{Inn}(H_1)$, and (4.4) gives:

$$\theta_t(a^i b^j c^k) a^l b^m c^n (a^i b^j c^k)^{-1} = a^l b^m c^{n+\epsilon i+\zeta j-ij+im-jl} = a^l b^m c^{n+(m-\zeta)i+(\epsilon-l)j}.$$

So the H_1-conjugacy classes of tH_1 have the form

$$[ta^l b^m c^n]_{H_1} = \{ta^l b^m c^{n+(m-\zeta)i+(\epsilon-l)j} \mid i, j \in \mathbb{Z}\}$$

and every such class contains an element of the form $ta^l b^m c^r$ for some r in the range $[0, \gcd(m-\zeta, l-\epsilon))$ (since we can choose appropriate values of i and j to achieve a power of c in the desired range). Thus the following set contains a representative for every H_1-conjugacy class in tH_1:

$$S = \{ta^l b^m c^r \mid l, m \in \mathbb{Z}, 0 \leq r < \gcd(m-\zeta, l-\epsilon)\}.$$

We claim that the length of each element of S is within a globally bounded distance of a minimal-length conjugacy representative, and therefore the growth of S provides an asymptotic upper bound for the relative growth of the H_1-conjugacy classes of tH_1, and hence also the G-conjugacy growth of tH_1. By Corollary 4.2.4, and Lemma 2.2.11, this set has growth equivalent to $n^2 \log n$ for $l, m \leq n$, which is therefore an upper bound for the relative conjugacy growth of tH_1 in this case.

Now we prove the claim. We have

$$0 \leq r < \gcd(m-\zeta, l-\epsilon) \leq \min(|m-\zeta|, |l-\epsilon|)$$

$$\leq \min(|m| + |\zeta|, |l| + |\epsilon|) \leq \min(|m|, |l|) + \max(|\zeta|, |\epsilon|),$$

and so, for l, m, r as in the set S, we can write $a^l b^m c^r = a^l b^m c^{r_1+r_2}$ where $0 \leq r_1 \leq \min(|m|, |l|)$ and $0 \leq r_2 \leq \max(|\zeta|, |\epsilon|)$. By the proof of Proposition 4.3.2, there is a global $K > 0$ such that

$$|l| + |m| \leq |a^l b^m c^{r_1}| \leq |l| + |m| + K.$$

The bound on r_2 depends only on θ_t (since the integers ζ and ϵ depend only on θ_t), so for our fixed coset tH_1 there is a constant $L > 0$ such that

$$|l| + |m| \leq |a^l b^m c^r| \leq |l| + |m| + K + L.$$

Now by Lemma 2.2.11 there exists $M > 0$ such that *any* element of the form $ta^l b^m c^n$ (and hence the H_1-conjugacy class $[ta^l b^m c^n]_{H_1}$) has length at least $|l| + |m| - M$, and at most $|a^l b^m c^n| + M$. In particular we have

$$|l| + |m| - M \leq |ta^l b^m c^r| \leq |l| + |m| + K + L + M.$$

Thus each $ta^l b^m c^r \in S$ is within bounded length of being a minimal-length conjugacy representative, as claimed.

Case $d = 1$**,** $d_1 \neq 0$

From equation (4.5) we have $\alpha + \delta = 2 - d_1$, and so $\lambda = \frac{2-d_1 \pm \sqrt{d_1^2 - 4d_1}}{2}$. It is easily checked that the only integer values of d_1 for which $|\lambda| = 1$ are $d_1 \in \{1, 2, 3, 4\}$.

Thinking of H_1 embedded in $\mathbb{R}^3$, from equation (4.4), we see that the H_1-conjugacy class of h lies on a parametric surface given by

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x \\ y \end{pmatrix} = D_1 \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} l \\ m \end{pmatrix},\ z = n + f'(i,j),\ i, j \in \mathbb{Z} \right\}$$

where $f'(i,j) = -\alpha\gamma\frac{i(i-1)}{2} - \beta\delta\frac{j(j-1)}{2} + (\epsilon + i - l)(\gamma i + \delta j) - ij\beta\gamma - ij + im - \zeta(\alpha i + \beta j)$. By considering the projection onto the x–y plane, we see that there are at most $\det D_1 = d_1 \in \{1, 2, 3, 4\}$ distinct orbits on a given surface. No two surfaces can intersect (since if they did it would violate the fact that the map D_1 is injective). Thus at most $\sim n^2$ surfaces pass through any vertical axis in the n-ball (since $B(n) \cap tH$ behaves like $B(n) \cap H$). So there are at most $\sim n^2$ surfaces in $B(n) \cap tH$, and hence at most $\sim n^2$ conjugacy classes. Therefore the relative conjugacy growth of tH_1 is $\mathcal{O}(n^2)$ in this case.

Case $d = -1$

In this case we have $f(0,0) = -2k$. If $x = c^k$ for some $k \in \mathbb{Z}$ we have $\theta_t(x)hx^{-1} = a^l b^m c^{n-2k}$. So conjugating only by elements $c^k \in H_1$ already reduces to just two conjugacy classes at each $ta^l b^m \langle c \rangle$, and further conjugation will only reduce this. Therefore the following is a sufficient set of representatives for the H_1-conjugacy classes in tH_1:

$$\left\{ta^l b^m c^\epsilon \mid l, m \in \mathbb{Z}, \epsilon \in \{0,1\}\right\}.$$

An analogous argument to that made for the set S above shows that the elements of this set are within bounded length of minimal-length conjugacy geodesics. Thus in this case the contribution to the conjugacy growth from the coset tH_1 is at most equivalent to the relative growth of this set, which is $\mathcal{O}(n^2)$.

In summary, cosets tH_1 where $\theta_t \in \mathrm{Inn}(H_1)$ have relative conjugacy growth $\mathcal{O}(n^2 \log n)$, and cosets where $\theta_t \in \mathrm{Aut}(H_1)\backslash\mathrm{Inn}(H_1)$ have relative conjugacy growth $\mathcal{O}(n^2)$. Thus the conjugacy growth of G is $\mathcal{O}(n^2 \log n)$. $\square$

The following corollary is proved in the same way as Corollary 4.3.4 above.

Corollary 4.5.3. *If G has a finite-index subgroup isomorphic to H_1, then G has transcendental conjugacy growth (with respect to any finite generating set).*

4.6 Free nilpotent groups

We end the chapter with a brief discussion of free nilpotent groups of class 2. Guba and Sapir [40] claim without proof that for any finitely generated nilpotent group, the conjugacy growth is bounded above by n^d where d is the Hirsch length. In this section we provide a counter-example.

Definition 4.6.1. Let F_r denote the free group of rank r. Then the *free nilpotent group* of rank r and class c is the quotient $F_r/F_r^{(c)}$. It has the presentation

$$N_{r,c} = \langle a_1, a_2, \ldots, a_r \mid [a_{i_0}, a_{i_1}, \ldots, a_{i_c}] = 1 \ \forall i_k \in \{1, 2, \ldots, r\}\rangle.$$

Note that the group $N_{2,2}$ is precisely the Heisenberg group H_1.

Proposition 4.6.2. *The free nilpotent group $N_{r,2}$ has conjugacy growth at least equivalent to $n^{r(r-1)}$.*

Proof. Since $[N_{r,2}, N_{r,2}]$ is central (and therefore abelian), the conjugacy growth of $N_{r,2}$ can be expressed as the sum of the standard growth of $Z := [N_{r,2}, N_{r,2}]$ (relative to $N_{r,2}$) and the conjugacy growth of $N_{r,2} \setminus Z$ (relative to $N_{r,2}$). Therefore the conjugacy growth of $N_{r,2}$ is bounded below by the standard growth of Z (relative to G).

We claim that Z has growth equivalent to $n^{r(r-1)}$. Observe that Z is free abelian of rank $\binom{r}{2}$, generated by the commutators of the form $[a_i, a_j]$ for $i \neq j$. By a similar argument to Lemma 4.1.6, each infinite cyclic factor $\langle [a_i, a_j] \rangle \leq Z$ has quadratic relative growth. Since there are $\binom{r}{2}$ such direct factors, and they all commute, Z has growth equivalent to $(n^2)^{\binom{r}{2}} = n^{r(r-1)}$ as claimed. $\square$

The Hirsch length, $h(N_{r,2}) = r + \binom{r}{2} = \frac{r^2+r}{2}$. For $r \geq 4$, $r(r-1) > \frac{r^2+r}{2}$ and so $c_{N_{r,2}}(n) \succcurlyeq n^{r(r-1)} \succ n^{\frac{r^2+r}{2}}$, which contradicts the claim of Guba and Sapir.

Chapter 5

Soluble Baumslag-Solitar groups

This chapter is based on joint work with Laura Ciobanu and Turbo Ho, appearing in the paper [11]. We study the conjugacy growth series of the soluble Baumslag-Solitar groups of the form $BS(1,k)$, defined below, with respect to their natural generating sets. In Section 5.1 we show that a certain subgroup $\mathbb{Z}_k$ has rational relative conjugacy growth. The complement of this subgroup is then shown to have transcendental relative conjugacy growth (Proposition 5.3.1) resulting in our main conclusion (Corollary 5.3.2) that the group as a whole has transcendental conjugacy growth series. We also provide explicit formulae for this growth series, and study the *growth rate* (see Section 2.2.1) showing in Corollary 5.3.3 that the conjugacy and standard growth rates are equal.

Throughout the chapter, we will write

$$G = BS(1,k) = \langle a, t \mid tat^{-1} = a^k \rangle$$

where $k \geq 2$ is a natural number. Lengths will always defined with respect to the symmetric generating set $\{a^{\pm 1}, t^{\pm 1}\}$. Let $\mathbb{Z}_k = \mathbb{Z}[\frac{1}{k}] = \{x \in \mathbb{Q} \mid k^n x \in \mathbb{Z} \text{ for some } n \in \mathbb{Z}\}$ and consider the semidirect product $\mathbb{Z}_k \rtimes \mathbb{Z}$, where the action of $\mathbb{Z}$ on $\mathbb{Z}_k$ is multiplication by k. Then $BS(1,k) \cong \mathbb{Z}_k \rtimes \mathbb{Z}$, with the isomorphism given by $a \to (1,0) \in \mathbb{Z}_k$ and $t \to (0,1) \in \mathbb{Z}$ where we write an element of G in the semidirect normal form (x,m).

Suppose that $m > 0$. Since

$$(t^m)^a = at^m a^{-1} = a \cdot a^{-k^m} t^m = (1 - k^m, m) \tag{5.1}$$

and $a^t = a^k$, we get that conjugation by generators amounts to:

$$(x, m)^a = (x + (1 - k^m), m) \text{ and } (x, m)^t = (kx, m). \tag{5.2}$$

The form of geodesics in the soluble Baumslag-Solitar groups has been studied in several articles, and we summarise here the results in a form convenient for further use. The following propositions are derived from section 4 of [17]. We restrict for now to only those elements with zero t-exponent sum.

Proposition 5.0.1. *Let $k = 2r + 1$ for some positive integer r. The set $\mathcal{E}_o$ of words in the following forms comprises a set of unique geodesic representatives for the elements of the subgroup $\mathbb{Z}_k$.*

(Oa.) $\{\epsilon, a^{\pm 1}, \ldots a^{\pm(r+1)}\}$

(Ob.) $\{a^{x_0} t a^{x_1} \cdots t a^{x_d} t^{-d} \mid d \geq 1, x_d \neq 0, \mathbf{A}\}$

(Oc.) $\{t^{-b} a^{x_0} t a^{x_1} \cdots t a^{x_d} t^{-c} \mid b, c, d \geq 1, d = b + c, x_0 \neq 0, x_d \neq 0, \mathbf{A}\}$

(Od.) $\{t^{-d} a^{x_0} t a^{x_1} \cdots t a^{x_d} \mid d \geq 1, x_0 \neq 0, \mathbf{A}\}$

Here $\mathbf{A}$ signifies the conditions $|x_d| \leq r + 1$, $|x_i| \leq r$ for $i < d$, and if $x_{d-1} = \pm r$ then $x_d \neq \mp 1$.

Proposition 5.0.2. *Let $k = 2r$ for some $r \geq 2$. The set $\mathcal{E}_e$ of words in the following forms comprise a set of unique geodesic representatives for the elements in $\mathbb{Z}_k$.*

(Ea.) $\{\epsilon, a^{\pm 1}, \ldots, a^{\pm(r+1)}\}$

(Eb.) $\{a^{x_0} t a^{x_1} \cdots t a^{x_d} t^{-d} \mid d \geq 1, x_d \neq 0, \mathbf{A}, \mathbf{B}\}$

(Ec.) $\{t^{-b} a^{x_0} t a^{x_1} \cdots t a^{x_d} t^{-c} \mid b, c, d \geq 1, d = b + c, x_0 \neq 0, x_d \neq 0, \mathbf{A}, \mathbf{B}\}$

(Ed.) $\{t^{-d} a^{x_0} t a^{x_1} \cdots t a^{x_d} \mid d \geq 1, x_0 \neq 0, \mathbf{A}\}$

Here, $\mathbf{A}$ signifies the conditions $|x_d| \leq r + 1$, and for each $0 \leq i < d$, $|x_i| \leq r$, if $x_{i-1} = r$ then $0 \leq x_i < r$ for $i < d$, and if $x_{i-1} = -r$ then $-r < x_i \leq 0$. And $\mathbf{B}$ signifies that the following subwords are forbidden: $a^{\pm r} t a^{\pm(r-2)} t a^{\mp 1} t^{-1}$, $a^{\pm(r-1)} t a^{\mp 1} t^{-1}$.

Proposition 5.0.3. *Let $k = 2$, i.e. $G = BS(1,2)$. The set $\mathcal{E}_2$ of words in the following forms comprise a set of unique geodesic representatives for the elements in $\mathbb{Z}_k$.*

(2a.) $\{\epsilon, a^{\pm 1}, a^{\pm 2}, a^{\pm 3}\}$

(2b.) $\{a^{x_0}ta^{x_1}t\cdots ta^{x_d}t^{-d} \mid d \geq 1, |x_d| \in \{2,3\}, \mathbf{A}\}$

(2c.) $\{t^{-b}a^{x_0}ta^{x_1}\cdots ta^{x_d}t^{-c} \mid b,c,d \geq 1, d = b+c, x_0 \neq 0, |x_d| \in \{2,3\}, \mathbf{A}\}$

(2d.) $\{t^{-d}a^{x_0}t\cdots ta^{x_d} \mid d \geq 1, x_0 \neq 0, \mathbf{A}\}$

Here, $\mathbf{A}$ signifies the conditions $|x_i| \leq 1$ for $i < d$, if $x_{i-1} \neq 0$ then $x_i = 0$ for $i < d$, if $x_d > 0$ then $x_{d-1} \geq 0$, and if $x_d < 0$ then $x_{d-1} \leq 0$.

5.1 The conjugacy classes $[(x,0)]$ in $BS(1,k)$

In this section we show that the conjugacy growth series of the subgroup $\mathbb{Z}_k$, relative to $G = BS(1,k)$, is rational with respect to the generating set $\{a,t\}$. We explicitly calculate the series via context-free grammars, and extract the growth rate. We note that Freden, Knudson, and Schofield [32] also use context-free grammars to calculate growth series, but they are concerned primarily with the so-called *horocyclic subgroup*, $\langle a\rangle$ in our notation, which is a proper subgroup of $\mathbb{Z}_k$. In fact, $\mathbb{Z}_k$ is the normal closure of $\langle a\rangle$.

We treat the cases of odd and even k separately.

5.1.1 Odd case

Let $k = 2r+1$ for some integer $r \geq 1$.

Proposition 5.1.1. *In $BS(1,2r+1)$ the set of words*

$$\mathcal{C}_o = \{\epsilon, a^{\pm 1}, \ldots a^{\pm(r+1)}\} \cup \{a^{x_0}ta^{x_1}t\cdots ta^{x_d}t^{-d} \mid d \geq 1, x_0 \neq 0, x_d \neq 0, \mathbf{A}\},$$

where $\mathbf{A}$ signifies the conditions $|x_d| \leq r+1$, $|x_i| \leq r$ for $i < d$, and if $x_{d-1} = \pm r$ then $x_d \neq \mp 1$, comprises a set of unique geodesic representatives for the conjugacy classes of G that lie in $\mathbb{Z}_k$.

Proof. Let $\mathcal{E}_o$ be as in Proposition 5.0.1 and note that $\mathcal{C}_o \subset \mathcal{E}_o$. We use the following key observation: if an element is represented by a word in $\mathcal{E}_o \setminus \mathcal{C}_o$, then it cannot be represented by a word in $\mathcal{C}_o$, by the uniqueness condition on $\mathcal{E}_o$. We will first prove that no pair of words in $\mathcal{C}_o$ represent the same conjugacy class, and then prove that every word in $\mathcal{E}_o$ is conjugate to a word in $\mathcal{C}_o$ with at most the same length. Then since every group element is represented in $\mathcal{E}_o$, every conjugacy class is represented (uniquely) in $\mathcal{C}_o$. Furthermore, this unique representative has length at most that of each of the corresponding (element-minimal) representatives in $\mathcal{E}_o$. This proves the proposition.

Proposition 5.0.1 implies that no two words in $\mathcal{C}_o$ represent equal elements. We show that no two words represent conjugate group elements either. Suppose, on the contrary, that $w, v \in \mathcal{C}_o$ represent conjugate elements. So there exists a non-zero integer m such that $t^m w t^{-m} =_G v$ (since a commutes with every element of $\mathbb{Z}_k$, and hence with every element of $\mathcal{C}_o$). First suppose that $w = a^n$ for $|n| \leq r+1$. Then $t^m a^n t^{m-1}$ is a word in either (Ob.) (with $x_0 = 0$) or (Oc.), depending on the sign of m, and thus by the above observation the word $v \notin \mathcal{C}_o$, which is a contradiction. Now suppose that $w = a^{x_0} t a^{x_1} t \cdots a^{x_d} t^{-d}$ for $d \geq 1$, $x_0 \neq 0$, with conditions **A**. So $v = t^m w t^{-m} = t^m a^{x_0} t a^{x_1} t \cdots a^{x_d} t^{-d-m}$. If $m > 0$, v is a word in (Ob.) (and not in $\mathcal{C}_o$). If $m < 0$, v is a word in either (Oc.) or (Od.). In both cases $v \notin \mathcal{C}_o$, which is again a contradiction.

Now let $w \in \mathcal{E}_o$. We show that there exists $v \in \mathcal{C}_o$ such that w and v represent conjugate group elements, and moreover $|w| \geq |v|$ (as words). We assume that $w \notin \mathcal{C}_o$ (otherwise the claim is trivial). First, suppose that w is in form (Ob.), and let $i > 0$ be such that x_i is the left-most non-zero power of a. Then the word $v = a^{x_i} t a^{x_{i+1}} t \cdots t a^{x_d} t^{-d+i}$ is in $\mathcal{C}_o$ and represents a conjugate of $\overline{w}$. Further, the number of $a^{\pm 1}$s in v is the same as that in w, and the number of $t^{\pm 1}$s in v is $(d-i)+(d-i) < 2d$ and therefore $|v| < |w|$. Now suppose w is of the form (Oc.) (resp. (Od.)). Let $v = a^{x_0} t a^{x_1} t \ldots a^{x_d} t^{-d}$. Since v is a leftward cyclic permutation of w by b (resp. d) places, the words represent conjugate elements and are of equal length. $\square$

Proposition 5.1.2. *Let $k = 2r+1$, where $r \geq 1$.*

1. *In $BS(1,k)$ the set $\mathcal{C}_o$ is unambiguous context-free.*

2. *The subgroup $\mathbb{Z}_k$ has rational relative conjugacy growth.*

Proof. (1) We show the language is context-free by exhibiting an explicit grammar. We use capital letters for variables and lower case for terminals. Write $a^{\pm n}$ as shorthand for the concatenation of n copies of the terminal $a^{\pm 1}$. It is straightforward to see that the following context-free grammar, starting from S, produces the set in question unambiguously. Each production rule either replaces a variable with an appropriate power of a, or adds corresponding instances of t and t^{-1}, together with appropriate powers of a (including a^0). If a^r or a^{-r} are produced, restrictions apply via V and W.

$$S \to \epsilon \mid A \mid T, \quad A \to a^{-r-1} \mid \cdots \mid a^{-1} \mid a \mid \cdots \mid a^{r+1}$$

$$B \to a^{-r+1} \mid \cdots \mid a^{-1} \mid a \mid \cdots \mid a^{r-1}, \quad T \to BtUt^{-1} \mid a^r tVt^{-1} \mid a^{-r}tWt^{-1}$$

$$U \to A \mid tUt^{-1} \mid T, \quad V \to tUt^{-1} \mid T \mid a^{-r-1} \mid \cdots \mid a^{-2} \mid a \mid \cdots \mid a^{r+1}$$

$$W \to tUt^{-1} \mid T \mid a^{-r-1} \mid \cdots \mid a^{-1} \mid a^2 \mid \cdots \mid a^{r+1}.$$

(2) By Theorem 2.5.2 the growth series of the language $\mathcal{C}_o$, and hence the relative conjugacy growth series of the subgroup $\mathbb{Z}_k$, is algebraic. However, a stronger result holds here. Applying the DSV method (see section 2.5) to the grammar above gives the growth series of the language $\mathcal{C}_o$. The production rules become the equations:

$$S(z) = 1 + A(z) + T(z), \quad A(z) = 2\sum_{i=1}^{r+1} z^i = 2\frac{z - z^{r+2}}{1-z},$$

$$B(z) = 2\sum_{i=1}^{r-1} z^i = 2\frac{z - z^r}{1-z}, \quad T(z) = B(z)U(z)z^2 + V(z)z^{r+2} + W(z)z^{r+2},$$

$$U(z) = A(z) + U(z)z^2 + T(z), \quad V(z) = U(z)z^2 + T(z) + 2\sum_{i=1}^{r+1} z^i - z,$$

$$W(z) = U(z)z^2 + T(z) + 2\sum_{i=1}^{r+1} z^i - z.$$

Solving these equations for $S(z)$ we find that

$$S_o(z) = S(z) = \frac{2z^{r+6} - 2z^{r+5} - 4z^{r+4} + 2z^{r+2} + 3z^3 + z^2 - z - 1}{z^3 - 2z^{r+3} + z^2 + z - 1}. \tag{5.3}$$

$\square$

Example 5.1.3. *For illustrative purposes, we derive an element of $\mathcal{C}_o$ using the grammar described above. The table below lists, on the left, the production rules used to produce the word $at^2a^rta^{-2}t^{-3}$, and the result of applying each rule on the right.*

production rule	*result*
$S \to T$	T
$T \to BtUt^{-1}$	$BtUt^{-1}$
$B \to a$	$atUt^{-1}$
$U \to tUt^{-1}$	at^2Ut^{-2}
$U \to T$	at^2Tt^{-2}
$T \to a^rtVt^{-1}$	$at^2a^rtVt^{-3}$
$V \to a^{-2}$	$at^2a^rta^{-2}t^{-3}$

Corollary 5.1.4. *The conjugacy classes in $\mathbb{Z}_k$, for $k = 2r+1$, have growth rate in the range $(\frac{4}{3}, 2)$.*

Proof. Denote by d_o the denominator of $S(z)$ in (5.3), that is, $d_o(z) = z^3 - 2z^{r+3} + z^2 + z - 1 = z^3(1 - z^r) + z(1 - z^{r+2}) + (z^2 - 1)$, which implies that for $z \in (-1, 0)$, $d_o(z) < 0$. Also, $d_o(\frac{1}{2}) = -\frac{1}{8} - \frac{1}{2^{r+2}} < 0$ and $d_o(\frac{3}{4}) = \frac{47}{64} - \frac{27}{32}(\frac{3}{4})^r > 0$, so there is a root $\rho_o \in (\frac{1}{2}, \frac{3}{4})$ of d_o. Furthermore, $d_o(0) = -1$ and $d_o'(z) > 0$ for $z \in [0, \frac{1}{2}]$, so ρ_o is the real root with smallest absolute value.

Write $a = \rho_0$ for ease of notation. The fact that a is a root of the denominator gives $2a^{r+3} = a^3 + a^2 + a - 1$. Using this identity we can substitute each $a^{\geq r}$ by the appropriate expression into the numerator and obtain $2a^{r+6} - 2a^{r+5} - 4a^{r+4} + 2a^{r+2} + 3a^3 + a^2 - a - 1 = a^7 - 2a^5 - a^4 + a^3 + 2a^2 - 1$. Furthermore, $a^7 - 2a^5 - a^4 + a^3 + 2a^2 - 1 = 0$ only for $a = \pm 1$, which is not the case, as $a \in (\frac{1}{2}, \frac{3}{4})$. Thus ρ_o is not a root of the numerator of $S(z)$ in (5.3), so the growth rate, which is the reciprocal of ρ_o, lies in the given range. $\square$

5.1.2 Even case

Let $k = 2r$, for some integer $r \geq 2$.

Proposition 5.1.5. *In $G = BS(1, 2r)$, $r \geq 2$, the set of words*

$$\mathcal{C}_e = \{\epsilon, a^{\pm 1}, \ldots, a^{\pm(r+1)}\} \cup \{a^{x_0}ta^{x_1}\cdots a^{x_d}t^{-d} \mid d \geq 1, \mathbf{A}, \mathbf{B}, x_0 \neq 0\}$$

*comprises a set of unique geodesic representatives for the conjugacy classes of G that lie in $\mathbb{Z}_k$. Here, **A** signifies the conditions $|x_d| \leq r+1$, and for each $0 \leq i < d$, $|x_i| \leq r$, if $x_{i-1} = r$ then $0 < x_i < r$ for $i < d$, and if $x_{i-1} = -r$ then $-r < x_i < 0$. And **B** signifies that the following subwords are forbidden: $a^{\pm r}ta^{\pm(r-2)}ta^{\mp 1}t^{-1}$, $a^{\pm(r-1)}ta^{\mp 1}t^{-1}$.*

Proof. Let $\mathcal{E}_e$ be as in Proposition 5.0.2 and note that $\mathcal{C}_e \subset \mathcal{E}_e$. We use the following key observation: if an element is represented by a word in $\mathcal{E}_e \setminus \mathcal{C}_e$, then it cannot be represented by a word in $\mathcal{C}_e$, by the uniqueness condition on $\mathcal{E}_e$. We will first prove that no pair of words in $\mathcal{C}_e$ represent the same conjugacy class, and then prove that every word in $\mathcal{E}_e$ is conjugate to a word in $\mathcal{C}_e$ with at most the same length. Then since every group element is represented in $\mathcal{E}_e$, every conjugacy class is represented (uniquely) in $\mathcal{C}_e$. Furthermore, this unique representative has length at most that of each of the corresponding (element-minimal) representatives in $\mathcal{E}_e$. This proves the proposition.

Proposition 5.0.2 implies that no two words in $\mathcal{C}_e$ represent equal elements. We show that no two words represent conjugate group elements either. Suppose, on the contrary, that $w, v \in \mathcal{C}_e$ represent conjugate elements. So there exists a non-zero integer m such that $t^m w t^{-m} =_G v$. First suppose that $w = a^n$ for $|n| \leq r+1$. Then $t^m a^n t^{m-1}$ is a word in either (Eb.) (with $x_0 = 0$) or (Ec.), depending on the sign of m, and thus by the above observation the word $v \notin \mathcal{C}_e$, which is a contradiction. Now suppose that $w = a^{x_0}ta^{x_1}t\cdots a^{x_d}t^{-d}$ for $d \geq 1$, $x_0 \neq 0$, with conditions **A** and **B**. So $v = t^m w t^{-m} = t^m a^{x_0}ta^{x_1}t\cdots a^{x_d}t^{-d-m}$. If $m > 0$, v is a word in (Eb.) (and not in $\mathcal{C}_e$). If $m < 0$, v is a word in either (Ec.) or (Ed.). In both cases, we have $v \notin \mathcal{C}_e$, which is again a contradiction.

Now let $w \in \mathcal{E}_e$. We claim that there exists $v \in \mathcal{C}_e$ such that w and v represent conjugate group elements, and moreover $|w| \geq |v|$ (as words). We assume that

$w \notin \mathcal{C}_e$ (otherwise the claim is trivial).

There are two exceptional cases. First, suppose

$$w = t^{-d} a^{x_0} t a^{x_1} \cdots a^{x_{d-2}} t a^{\pm(r-1)} t a^{\mp 1}$$

with $d \geq 1$ and conditions **A** (so in particular w is in the form (Ed.)). Then $\overline{w}$ is conjugate to the element represented by $a^{x_0} t a^{x_1} \cdots a^{x_{d-1}} t a^{\pm(r-1)} t a^{\mp 1} t^{-d}$. This word contains a forbidden subword and therefore does not satisfy condition **B**, so is not in $\mathcal{C}_e$. However, it represents the same element as $v := a^{x_0} t a^{x_1} \cdots a^{x_{d-1}} t a^{\mp(r+1)} t^{-d+1} \in \mathcal{C}_e$. We also have $|w| = \sum_{i=0}^{d-2} x_i + (r-1) + 1 + 2d > \sum_{i=0}^{d-2} x_i + (r+1) + 2(d-1) = |v|$.

For the second exceptional case, suppose

$$w = t^{-d} a^{x_0} t a^{x_1} \cdots a^{x_{d-3}} t a^{\pm r} t a^{\pm(r-2)} t a^{\mp 1}.$$

Then $\overline{w}$ is conjugate to the element represented by

$$a^{x_0} t a^{x_1} \cdots a^{x_{d-3}} t a^{\pm r} t a^{\pm(r-2)} t a^{\mp 1} t^{-d},$$

which contains a forbidden subword, but represents the same element as $v := a^{x_0} t a^{x_1} \cdots a^{x_{d-3}} t a^{\mp r} t a^{\mp(r+1)} t^{-d+1} \in \mathcal{C}_e$. In this case we have

$$|w| = \sum_{i=0}^{d-3} x_i + r + (r-2) + 1 + 2d = \sum_{i=0}^{d-3} x_i + r + (r+1) + 2(d-1) = |v|.$$

For the general case, where w is in the form (Eb.), (Ec.), or (Ed.) (excluding the exceptional cases) and is not already an element of $\mathcal{C}_e$, it is clear that conjugation by $t^{\pm 1}$ an appropriate number of times takes w to a word in $\mathcal{C}_e$, which has at most the same length as w. $\square$

Proposition 5.1.6. *Let $k = 2r$, $r \geq 2$.*

1. *In $G = BS(1, k)$, the set $\mathcal{C}_e$ is an unambiguous context-free language.*

2. *The subgroup $\mathbb{Z}_k$ has rational conjugacy growth.*

Proof. (1) We claim that the following grammar, with S as the starting point, generates $\mathcal{C}_e$ unambiguously.

$$S \to \epsilon \mid A \mid T, \quad A \to a^{-(r+1)} \mid a^{-r} \mid \cdots \mid a^{-1} \mid a \mid \cdots \mid a^{r+1}$$

$$T \to BtUt^{-1} \mid a^{r}tVt^{-1} \mid a^{-r}tWt^{-1} \mid a^{r-1}tXt^{-1} \mid a^{-(r-1)}tYt^{-1}$$

$$B \to a^{-(r-2)} \mid a^{-(r-3)} \mid \cdots \mid a^{-1} \mid a \mid \cdots \mid a^{r-2}, \quad U \to tUt^{-1} \mid T$$

$$V \to a \mid a^2 \mid \cdots \mid a^{r-1} \mid tUt^{-1} \mid atUt^{-1} \mid \cdots \mid a^{r-3}tUt^{-1} \mid a^{r-2}tXt^{-1} \mid a^{r-1}tXt^{-1}$$

$$W \to a^{-1} \mid a^{-2} \mid \cdots \mid a^{-(r-1)} \mid tUt^{-1} \mid a^{-1}tUt^{-1} \mid \cdots$$

$$\cdots \mid a^{-(r-3)}tUt^{-1} \mid a^{-(r-2)}tYt^{-1} \mid a^{-(r-1)}tYt^{-1}$$

$$X \to a^{-(r+1)} \mid a^{-r} \mid \cdots \mid a^{-2} \mid a \mid a^2 \mid \cdots \mid a^{r+1} \mid U$$

$$Y \to a^{-(r+1)} \mid a^{-r} \mid \cdots \mid a^{-2} \mid a^{-1} \mid a^2 \mid \cdots \mid a^{r+1} \mid U$$

Starting from S, this grammar produces words in $\mathcal{C}_e$ by choosing the values of the powers x_i from left to right, while keeping track of the number d of such powers. If x_i is chosen to be $\pm r$ or $\pm(r-1)$, restrictions apply to the following power.

(2) We use the grammar above to explicitly calculate the growth function. The

grammar yields the following system of equations.

$$S(z) = 1 + A(z) + T(z), \quad A(z) = 2\sum_{i=1}^{r+1} z^i = \frac{2(z - z^{r+2})}{1 - z},$$

$$T(z) = t^2 B(z)U(z) + 2z^{r+2}V(z) + 2z^{r+1}X(z),$$

$$B(z) = 2\sum_{i=1}^{r-2} z^i = \frac{2(z - z^{r-1})}{1 - z}, \quad U(z) = z^2U(z) + T(z),$$

$$V(z) = W(z) = \sum_{i=1}^{r-1} z^i + z^2U(z)\sum_{i=0}^{r-3} z^i + z^2X(z)(z^{r-2} + z^{r-1})$$

$$= \frac{z - z^r}{1 - z} + z^2U(z)\frac{1 - z^{r-2}}{1 - z} + z^2X(z)(z^{r-2} + z^{r-1}),$$

$$X(z) = Y(z) = 2\sum_{i=1}^{r+1} z^i - z + U(z) = \frac{2(z - z^{r+2})}{1 - z} - U(z).$$

Solving these for $S(z)$ yields the following rational expression:

$$S_e(z) = \frac{n(z)}{d(z)} = \frac{-1 - 2z^{r+2} + 2z^3 + z^4 + 2z^2 - 4z^{3r+6} + 4z^{3r+8} - 2z^{2r+8} + 4z^{3r+4} - 4z^{r+6}}{(2z^{2r+4} - 2z^{r+4} - z^3 + 2z^{r+2} - z^2 - z + 1)(z - 1)} \tag{5.4}$$

$$+ \frac{4z^{6+2r} - 2z^{2r+7} + 2z^{2r+2} + 2z^{r+5} - 6z^{2r+4} - 6z^{r+3} + 6z^{r+4}}{(2z^{2r+4} - 2z^{r+4} - z^3 + 2z^{r+2} - z^2 - z + 1)(z - 1)}.$$

That is, the denominator of $S(z)$ is $d(z) = (2z^{2r+4} - 2z^{r+4} - z^3 + 2z^{r+2} - z^2 - z + 1)(z - 1)$ and the numerator $n(z) = -1 + 2z^2 + 2z^3 + z^4 - 2z^{r+2} - 6z^{r+3} + 6z^{r+4} + 2z^{r+5} - 4z^{r+6} + 2z^{2r+2} - 6z^{2r+4} + 4z^{6+2r} - 2z^{2r+7} - 2z^{2r+8} + 4z^{3r+4} - 4z^{3r+6} + 4z^{3r+8}$. □

Corollary 5.1.7. *The conjugacy classes in $\mathbb{Z}_k$ have growth rate in the range $(\frac{4}{3}, 2)$.*

Proof. For $z \in [-\frac{1}{2}, 0]$, $d(z) = 2z^{2r+4} - 2z^{r+4} - z^3 + 2z^{r+2} - z^2 - z + 1 = (1 - z) - z^2(1 - z^{2r+2}) - z^3(1 - z^{2r+1}) + 2z^{r+2}(1 - z^2) \geq 1 - \frac{1}{4} - \frac{1}{16} > 0$, so there is no root in $[-\frac{1}{2}, 0]$. Similarly, for $z \in [-\frac{3}{4}, -\frac{1}{2}]$, we have that $d(z) = (1 - z) - z^2(1 - z^{2r+2}) - z^3(1 - z^{2r+1}) + 2z^{r+2}(1 - z^2) \geq \frac{3}{2} - \frac{4}{9} - 2(\frac{3}{4})^4 > 0$, so there is no root in $[-\frac{3}{4}, -\frac{1}{2}]$.

We also have $d(\frac{1}{2}) = \frac{1}{8} + \frac{1}{2^{2r+3}} - \frac{1}{2^{r+3}} + \frac{1}{2^{r+1}} > 0$ and $d(\frac{3}{4}) < 0$. So there is a root $\in (\frac{1}{2}, \frac{3}{4})$ of d. Furthermore, $d(0) = 1$ and $d'(z) < 0$ for $z \in [0, \frac{1}{2}]$, so the real root with smallest absolute value lies in $(\frac{1}{2}, \frac{3}{4})$.

Write a to be the real root with smallest absolute value of $d(z)$. The fact that a is a root of the denominator gives $2a^{2r+4} - 2a^{r+4} + 2a^{r+2} - a^3 - a^2 - a + 1 = 0$. In particular, $2a^{2r+4} = 2a^{r+4} - 2a^{r+2} + a^3 + a^2 + a - 1 = 0$ and $a^3 + a^2 + a - 1 = 2a^{2r+4} - 2a^{r+4} + 2a^{r+2}$. Using these identities we get that

$$\begin{aligned}
n(a) = & -1 + 2a^2 + 2a^3 + a^4 - 2a^{r+2} - 6a^{r+3} + 6a^{r+4} + 2a^{r+5} - 4a^{r+6+2a^{2r+2}} \\
& - 6a^{2r+4} + 4a^{6+2r} - 2a^{2r+7} - 2a^{2r+8} + 4a^{3r+4} - 4a^{3r+6} + 4a^{3r+8} \\
= & (a+1)(a^3 + a^2 + a - 1) - 2a^{r+2} - 6a^{r+3} + 6a^{r+4} + 2a^{r+5} - 4a^{r+6} + 2a^{2r+2} \\
& - 6a^{2r+4} + 4a^{6+2r} - 2a^{2r+7} - 2a^{2r+8} + 2a^{2r+4}(2a^r - 2a^{r+2} + 2a^{r+4}) \\
= & (a+1)(2a^{2r+4} - 2a^{r+4} + 2a^{r+2}) - 2a^{r+2} - 6a^{r+3} + 6a^{r+4} + 2a^{r+5} \\
& - 4a^{r+6} + 2a^{2r+2} - 6a^{2r+4} + 4a^{6+2r} - 2a^{2r+7} - 2a^{2r+8} \\
& + (2a^{r+4} - 2a^{r+2} + a^3 + a^2 + a - 1)(2a^r - 2a^{r+2} + 2a^{r+4}) \\
= & 2a^r(a+1)(a-1)^2((a^3 - a - 1)a^{r+2} + a^4 + a^2 - 1)
\end{aligned}$$

However, $a \in (\frac{1}{2}, \frac{3}{4})$ implies $a^3 - a - 1 < 0$ and $a^4 + a^2 - 1 < 0$, so $((a^3 - a - 1)a^{r+2} + a^4 + a^2 - 1) < 0$. Also $a \neq -1, 0, 1$, so a is not a root of the numerator of $S(z)$ in (5.4), and thus the growth rate, which is the reciprocal of a, lies in the given range.

$\square$

5.1.3 The case $k = 2$

Let $G = BS(1, 2)$.

Proposition 5.1.8. *In $BS(1, 2)$ the set of words*

$$\mathcal{C}_2 = \{\epsilon, a^{\pm 1}, a^{\pm 3}\} \cup \{a^{x_0} t a^{x_1} t \cdots t a^{x_d} t^{-d} \mid d \geq 1, |x_d| \in \{2, 3\}, x_0 \neq 0, \mathbf{A}\}$$

comprises a set of unique geodesic representatives for the conjugacy classes of G that lie in the subgroup $\mathbb{Z}_k$. Here, $\mathbf{A}$ signifies the conditions $|x_i| \leq 1$ for $i < d$, if $x_{i-1} \neq 0$ then $x_i = 0$ for $i < d$, if $x_d > 0$ then $x_{d-1} \geq 0$, and if $x_d < 0$ then $x_{d-1} \leq 0$.

Proof. Let $\mathcal{E}_2$ be as in Proposition 5.0.3 and note that $\mathcal{C} \subset \mathcal{E}_2$. As above, we use the following key observation: if an element is represented by a word in $\mathcal{E}_2 \setminus \mathcal{C}_2$, then it cannot be represented by a word in $\mathcal{C}_2$, by the uniqueness condition on $\mathcal{E}_2$. We will first prove that no pair of words in $\mathcal{C}_2$ can represent the same conjugacy class, and then prove that every word in $\mathcal{E}_2$ is conjugate to a word in $\mathcal{C}_2$ of at most the same length, proving the proposition.

We show that no pair of words in $\mathcal{C}_2$ represent conjugate elements. Let $w \in \mathcal{C}_2$ and suppose on the contrary that it represents the same conjugacy class as some $v \in \mathcal{C}_2$. Since no pair of words in $\mathcal{E}_2$ represent the same element, there exists $m \neq 0$ with $t^m w t^{-m} =_G v$. First consider the case where $w \in \{a^{x_0} t a^{x_1} t \cdots t a^{x_d} t^{-d} \mid d \geq 1, |x_d| \in \{2,3\}, x_0 \neq 0, A\}$. Then $t^m w t^{-m}$ has the form (2b.) with $x_0 \neq 0$, or (2c.), or (2d.), which contradicts the key observation. Now consider the case $w = a^{\pm 1}$, with $m = 1$. Then $twt^{-1} = ta^{\pm 1}t^{-1} =_G a^{\pm 2}$, and hence the word twt^{-1} cannot be in $\mathcal{C}_2$ by the uniqueness condition on $\mathcal{E}_2$. In the case $w = a^{\pm 1}$, with $m = -1$, we have $t^{-1}wt$ in the form (2d.), again a contradiction. Next, consider $w = a^{\pm 1}$ with $|m| \geq 2$. We have $t^m a^{\pm 1} t^{-m} =_G t^{m-1} a^{\pm 2} t^{-(m-1)}$, which is a word in the form (2b.) with $x_0 \neq 0$, or (2c.), or (2d.), again contradicting the key observation. Finally consider the case $w = a^{\pm 3}$. Then $t^m a^{\pm 3} t^{-m}$ is in the form (2b.) with $x_0 \neq 0$, or (2c.), or (2d.), again contradicting the key observation.

Now let $w \in \mathcal{E}_2$. We show that there exists $v \in \mathcal{C}_2$ such that w and v represent conjugate elements, and that $|w| \geq |v|$. We assume $w \notin \mathcal{C}_2$. Firstly, $a^{\pm 2}$ is conjugate to $a^{\pm 1} \in \mathcal{C}_2$, which has strictly shorter length. Now suppose $w = t^{-d} a^{x_0} t \cdots a^{x_{d-1}} ta \in$ (2d.), where $x_{d-1} \in \{0,1\}$. Then w is conjugate, via t^d, to $a^{x_0} t \cdots t a^{x_{d-1}} tat^d$, which has the same length. This word represents the same element as $a^{x_0} t \cdots t a^{x_{d-1}+2} t^{d-1} \in \mathcal{C}_2$ which has strictly smaller length. Similarly, if $w = t^{-d} a^{x_0} t \cdots a^{x_{d-1}} t a^{-1} \in$ (2d.), we must have $x_{d-1} \in \{-1, 0\}$, and w represents the same conjugacy class as the shorter word $a^{x_0} t \cdots t a^{x_{d-1}-2} t^{d-1} \in \mathcal{C}_2$. In all other cases, w is clearly conjugate, via an appropriate number of $t^{\pm 1}$s, to a word in $\mathcal{C}_2$ of equal or shorter length. □

Proposition 5.1.9. *The subgroup $\mathbb{Z}_2 \leq BS(1,2)$ has rational conjugacy growth.*

Proof. It is straightforward to see that the following grammar, starting from S, produces $\mathcal{C}_2$ unambiguously.

$$S \to \epsilon \mid A \mid T, \quad A \to a^{-3} \mid a^{-1} \mid a \mid a^3,$$

$$T \to at^2Ut^{-2} \mid a^{-1}t^2Ut^{-2} \mid ata^2t^{-1} \mid ata^3t^{-1} \mid a^{-1}ta^2t^{-1} \mid a^{-1}ta^3t^{-1},$$

$$U \to tUt^{-1} \mid T \mid a^{-3} \mid a^{-2} \mid a^2 \mid a^3$$

The grammar becomes the following system of equations.

$$S(z) = 1 + A(z) + T(z), \quad A(z) = 2(z + z^3),$$

$$T(z) = 2z^5U(z) + 2z^5 + 2z^6, \quad U(z) = z^2U(z) + T(z) + 2(z^2 + z^3).$$

Solving these yields the following rational expression:

$$S(z) = \frac{1 + 2z - z^2 - 2z^5 - 2z^6 + 2z^7 - 2z^8}{1 - z^2 - 2z^5}. \tag{5.5}$$

□

Corollary 5.1.10. *The conjugacy classes in* $\mathbb{Z}_2 \leq BS(1,2)$ *have growth rate approximately* 1.348.

Proof. The only real root of the polynomial $1 - z^2 - 2z^5$, the denominator of (5.5), is approximately 0.742. Denote this root by a, so that $1 - a^2 - 2a^5 = 0$. Using this identity, we find that the numerator of (5.5) is equal to $a + a^2 - a^4 + a^6$ when $z = a$. Since $a^4 < a^2$, we see that a is not a root of the numerator. Therefore the growth rate is the reciprocal of a, approximately 1.348. □

5.2 The conjugacy classes $[(x, m)]$, $m \neq 0$, in $BS(1, k)$

In this section we find and describe a set of minimal representatives for the conjugacy classes of the form $[(x, m)]$ with $m \neq 0$.

5.2.1 The conjugacy geodesics

We first need the following result by Collins-Edjvet-Gill, which although stated for k even, also holds for k odd.

Lemma 5.2.1. *[17, Lemma 2.2] Let w be a geodesic word. Then:*

1. *If w has a subword of the form $t^{-r}a^{i_0}ta^{i_1}t\cdots ta^{i_n}t^{-s}$, where $i_0, i_n \neq 0$, $r, s, n \geq 1$, then $r + s \leq n$.*
2. *If w has a subword of the form $t^r a^{i_0}t^{-1}a^{i_1}t^{-1}\cdots t^{-1}a^{i_n}t^s$, where $i_0, i_n \neq 0$, $r, s, n \geq 1$, then $r + s \leq n$.*
3. *w has at most one subword of the form $t^{-1}a^i t$ where $i \neq 0$, and at most one subword of the form $ta^i t^{-1}$ where $i \neq 0$.*

The following proposition shows that a conjugacy geodesic w has no 'pinches', that is, no subwords of the form $t^{-1}a^i t$ or $ta^i t^{-1}$ where $i \neq 0$.

Proposition 5.2.2. *Every conjugacy geodesic w for $[(x, m)]$ with $m > 0$ must be, up to a cyclic permutation, of the form $a^{x_0}ta^{x_1}t\cdots a^{x_{m-1}}t$ for some $x_0, \cdots, x_{m-1} \in \mathbb{Z}$.*

Proof. Let w be a conjugacy geodesic for $[(x, m)]$.

Suppose that w contains t^{-1} non-trivially. By Lemma 5.2.1 (3), after cyclically permuting w if necessary, we may assume that

$$w = a^{x_0}ta^{x_1}t\cdots a^{x_{n-1}}ta^{y_n}t^{-1}a^{y_{n-1}}t^{-1}\cdots a^{y_{m+1}}t^{-1}$$

with $x_0, y_n \neq 0$, and $n > m$. Since a commutes with any word with t-exponent sum equal to zero, we can rewrite w as follows, without increasing its length:

$$\begin{aligned} w &= a^{x_0}ta^{x_1}t\cdots ta^{x_{n-1}}(ta^{y_n}t^{-1})a^{y_{n-1}}t^{-1}a^{y_{n-2}}\cdots a^{y_{m+1}}t^{-1} \\ &= a^{x_0}ta^{x_1}t\cdots a^{x_{n-2}}(t^2a^{y_n}t^{-1}a^{x_{n-1}+y_{n-1}}t^{-1})a^{y_{n-2}}\cdots a^{y_{m+1}}t^{-1} \\ &\vdots \\ &= a^{x_0}ta^{x_1}t\cdots a^{x_m}t^{n-m}a^{y_n}t^{-1}a^{x_{n-1}+y_{n-1}}t^{-1}\cdots a^{x_{m+1}+y_{m+1}}t^{-1}. \end{aligned}$$

For ease of notation we will rename exponents so that

$$w = a^{x_0}ta^{x_1}t\cdots a^{x_m}t^{n-m}a^{y_n}t^{-1}a^{y_{n-1}}t^{-1}\cdots a^{y_{m+1}}t^{-1},$$

and note that its cyclic permutation

$$a^{x_1}t\cdots a^{x_m}t^{n-m}a^{y_n}t^{-1}a^{y_{n-1}}t^{-1}\cdots a^{y_{m+1}}t^{-1}a^{x_0}t$$

has a subword $t^{n-m}a^{y_n}t^{-1}a^{y_{n-1}}t^{-1}\cdots a^{y_{m+1}}t^{-1}a^{x_0}t$ which contradicts Lemma 5.2.1 (2). So, w cannot contain any t^{-1}.

Thus, w must have the form $a^{x_0}ta^{x_1}t\cdots a^{x_{m-1}}t$, up to a cyclic permutation. □

Now by checking through the list of geodesics in [17, Section 4], we see a conjugacy geodesic must be of the form (MWe1a). Translating this to our language and using the fact that a cyclic permutation of a conjugacy geodesic is still a geodesic, we obtain the following proposition:

Proposition 5.2.3. *In $BS(1,k)$, every conjugacy geodesic w for $[(x,m)]$ with $m > 0$ must be, up to a cyclic permutation, of the form $a^{x_0}ta^{x_1}t\cdots a^{x_{m-1}}t$ for some $x_0,\cdots,x_{m-1} \in \mathbb{Z}$ such that:*

- *If $k = 2r+1$ is odd, then $|x_i| \leq r$ for every i.*
- *If $k = 2r$ is even, then $|x_i| \leq r$, and for each i, if $x_{i-1} = r$ then $0 \leq x_i < r$, and if $x_{i-1} = -r$ then $-r < x_i \leq 0$. (Here and henceforth in this section, we use the convention that $x_{-1} = x_{m-1}$.)*

5.2.2 The conjugacy representatives

We now give conjugacy representatives for a fixed $m > 0$. Recall that by 5.2 two elements (x,m) and (y,n) are conjugate only if $m = n$, so it suffices to restrict the analysis to elements of the form (x,m), with m fixed, in the following arguments.

Lemma 5.2.4. *Suppose $k = 2r+1$ and $m > 0$. Let*

$$\mathcal{A}_m = \{a^{x_0}ta^{x_1}t\cdots a^{x_{m-1}}t \mid |x_i| \leq r\} \setminus \{(a^{-r}t)^m\}.$$

Then two words in $\mathcal{A}_m$ are conjugate if and only if they are cyclic permutations of each other, and every word in $\mathcal{A}_m$ is a conjugacy geodesic.

The proof of the odd case is the same as the proof for even case, but simpler. Thus, we shall only prove the even case.

Lemma 5.2.5. *Suppose $k = 2r$ and $m > 0$. Let $\mathcal{A}_m$ be the set of words $a^{x_0}ta^{x_1}t\cdots a^{x_{m-1}}t$ satisfying*

1. *$|x_i| \leq r$,*
2. *for each i, if $x_{i-1} = r$ then $0 \leq x_i < r$, and if $x_{i-1} = -r$ then $-r < x_i \leq 0$,*
3. *if m is even, $(a^{-(r-1)}ta^{-r}t)^{\frac{m}{2}}$ and $(a^{-r}ta^{-(r-1)}t)^{\frac{m}{2}}$ are excluded from $\mathcal{A}_m$.*

Then two words in $\mathcal{A}_m$ are conjugate if and only if they are cyclic permutations of each other, and every word in $\mathcal{A}_m$ is a conjugacy geodesic.

Proof. We first show that two distinct words, $a^{x_0}ta^{x_1}t\cdots a^{x_{m-1}}t$ and $a^{y_0}ta^{y_1}t\cdots a^{y_{m-1}}t$, in $\mathcal{A}_m$ cannot be conjugate by a^ℓ, $\ell \neq 0$. Suppose we have such a pair, and suppose these two words represent the elements (x, m) and (y, m). Since $x = \sum_{i=0}^{m-1} x_i k^i$, (1) implies that $|x| \leq (k^m - 1)\frac{k}{2(k-1)}$, and similarly $|y| \leq (k^m - 1)\frac{k}{2(k-1)}$. The conjugation by a^ℓ translates into $x = y + \ell(k^m - 1)$, which together with the above inequalities forces $|\ell| = 1$. Without loss of generality, we will assume that $\ell = 1$.

Now as $\ell = 1$, $x - y = \sum_{i=0}^{m-1}(x_i - y_i)k^i = k^m - 1 = (k-1) + (k-1)k + \cdots + (k-1)k^{m-1}$. Suppose $x_i - y_i \neq k - 1$ for some i, and let i_1 be the smallest such index. By taking

$$\sum_{i=0}^{m-1}(x_i - y_i)k^i = (k-1) + (k-1)k + \cdots + (k-1)k^{m-1} \tag{5.6}$$

modulo k^{i_1+1}, we must have $x_{i_1} - y_{i_1} = -1$ since all higher terms are $0 \pmod{k^{i_1+1}}$, all lower terms on both sides cancel, and so $(x_{i_1} - y_{i_1})k^{i_1} \equiv (k-1)k^{i_1} \pmod{k^{i_1+1}}$ (and $|x_{i_1} - y_{i_1}| \leq k$). By taking equation (5.6) modulo k^{i_1+2}, similar computations show that $x_{i_1+1} - y_{i_1+1} \equiv 0 \pmod{k}$. If $x_{i_1+1} - y_{i_1+1} = 0$, then the same argument implies $x_{i_1+2} - y_{i_1+2} \equiv 0 \pmod{k}$, etc. Suppose i_2 is the first index such that $x_{i_2} - y_{i_2} \neq 0$.

- If $x_{i_2} - y_{i_2} = k$, this forces $x_{i_2} = r$ and $y_{i_2} = -r$. By (2), this means $0 \leq x_{i_2+1} < r$ and $0 \geq y_{i_2+1} > -r$, so $0 \leq x_{i_2+1} - y_{i_2+1} \leq k-2$. But equation (5.6) modulo k^{i_2+2} implies $x_{i_2+1} - y_{i_2+1} \equiv -1 \pmod{k}$, a contradiction.

- If $x_{i_2} - y_{i_2} = -k$, this forces $x_{i_2} = -r$ and $y_{i_2} = r$. By (2), this means $0 \geq x_{i_2+1} > -r$ and $0 \leq y_{i_2+1} < r$, so $0 \geq x_{i_2+1} - y_{i_2+1} \geq -k+2$. But equation (5.6) modulo k^{i_2+2} implies $x_{i_2+1} - y_{i_2+1} \equiv 1 \pmod{k}$, a contradiction.

This means that we actually have $x_i - y_i = k-1 = 2r-1$ for all i. Thus, $(x_i, y_i) = (r, -r+1)$ or $(r-1, -r)$ for every i. But by (2), m must be even since the r and $r-1$ need to alternate in w as a cyclic word, and $a^{y_0}ta^{y_1}t \cdots a^{y_{m-1}}t = (a^{-r}ta^{-r+1}t)^{\frac{m}{2}}$ or $(a^{-r+1}ta^{-r}t)^{\frac{m}{2}}$. This violates (3), and thus any two distinct words in $\mathcal{A}_m$ cannot be conjugate by a^{ℓ}.

Now let $w = a^{x_0}ta^{x_1}t \cdots a^{x_{m-1}}t \in \mathcal{A}_m$ and suppose some word u in $\mathcal{A}_m$ is conjugate to w, that is, $w = u^{(x,l)}$, where $(x,l) \in BS(1,k)$. Consider the cyclic permutation w' of w ending in t given by $w' = w^v$, where $v = a^{x_0}ta^{x_1}t \cdots a^{x_{l-1}}t$ and $x_p = x_{p \bmod m}$, for any $p \in \mathbb{N}$. Clearly w' is also in $\mathcal{A}_m$, and v has the form (y,l). Then $u^{(x,l)^{-1}(y,l)} = w'$, so u is conjugate to w' by a power of a since $(x,l)^{-1}(y,l) = (z,0)$ for some $z \in \mathbb{Z}_k$, which gives a contradiction to our previous claim. Thus u must be a cyclic permutation of w, proving the first assertion of the lemma.

Suppose $w = a^{x_0}ta^{x_1}t \cdots a^{x_{m-1}}t \in \mathcal{A}_m$, and take a conjugacy geodesic u of $[w]$. By Proposition 5.2.3, u is a word in $\mathcal{A}_m$ if it is not excluded by (3) and by the first assertion of this lemma, w is a cyclic permutation of u, thus also a conjugacy geodesic. If u is of the form described in (3), then by the proof above, $w = (a^{r-1}ta^rt)^{\frac{m}{2}}$ or $(a^rta^{r-1}t)^{\frac{m}{2}}$ and has the same length as u, so is also a conjugacy geodesic. □

The above discussion concerns the case when $m > 0$. The antiautomorphism $(x,m) \mapsto (x,m)^{-1} = (-\frac{x}{k^m}, -m)$ provides a bijection between elements of the form (x,m) and those of the form $(y,-m)$. Since g^{-1} has the same length as g, and taking inverses preserves conjugacy, the results above translate to the case when $m < 0$. Thus, writing $\mathcal{A}_+ = \bigcup_{m>0} \mathcal{A}_m$ and $\mathcal{A}_- = \mathcal{A}_+^{-1}$, we have the following description of conjugacy representatives:

Corollary 5.2.6. *The set $\mathcal{A}$, modulo cyclic permutations, gives a set of minimal length conjugacy representatives for the conjugacy classes of the group $BS(1,k)$ that are not in the base group $\mathbb{Z}_k$.*

1. *Let $k = 2r+1$, $r \geq 1$. Then $\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_-$, where*

$$\mathcal{A}_+ = \{a^{x_0}ta^{x_1}t\cdots a^{x_{m-1}}t \mid m \geq 1, |x_i| \leq r \text{ for } 0 \leq i \leq m-1\} \setminus \{(a^{-r}t)^m \mid m \geq 1\},$$

$$\mathcal{A}_- = \{t^{-1}a^{x_0}t^{-1}a^{x_1}\cdots t^{-1}a^{x_{m-1}} \mid m \geq 1, |x_i| \leq r\} \setminus \{(t^{-1}a^{-r})^m \mid m \geq 1\}.$$

2. *Let $k = 2r$, $r \geq 1$. Then $\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_-$, where*

$$\mathcal{A}_+ = \{a^{x_0}ta^{x_1}t\cdots a^{x_{m-1}}t \mid m \geq 1, |x_i| \leq r, \forall i(x_{i-1} = \pm r \implies 0 \leq \pm x_i < r)\}$$

$$\setminus \{(a^{-r+1}ta^{-r}t)^{\frac{m}{2}}, (a^{-r}ta^{-r+1}t)^{\frac{m}{2}} \mid m \geq 2, m \equiv 0 \pmod 2\},$$

$$\mathcal{A}_- = \{t^{-1}a^{x_0}t^{-1}\cdots t^{-1}a^{x_{m-1}} \mid m \geq 1, |x_i| \leq r, \forall i(x_i = \pm r \implies 0 \leq \pm x_{i-1} < r)\}$$

$$\setminus \{(t^{-1}a^{r}t^{-1}a^{r-1})^{\frac{m}{2}}, (t^{-1}a^{r-1}t^{-1}a^{r})^m \mid m \geq 2, m \equiv 0 \pmod 2\}.$$

Proof. We have shown that the elements of $\mathcal{A}$ are conjugacy geodesics, unique up to cyclic permutation. It remains to show that every conjugacy class of $BS(1,k)$ not contained in $\mathbb{Z}_k$ has a representative in $\mathcal{A}$.

By the observation above, we only need to show that for $m > 0$, every element of the form (x,m) is conjugate to an element represented by a word in $\mathcal{A}_+$. Again, we will only prove this for the more complicated case of $k = 2r$.

First, we show that any element of the form (x,m) is conjugate to an element represented by a word of the form $a^{x_0}ta^{x_1}t\cdots a^{x_{m-1}}ta^{x_m}$ (where $x_i \in \mathbb{Z}$). From [17], the element (x,m) has a (geodesic) representative in one of the following forms:

- MWe1a: $a^{x_0}ta^{x_1}t\cdots ta^{x_m}$
- MWe2a: $t^{-n}a^{x_0}ta^{x_1}t\cdots ta^{x_{m+n}}$, some $1 \leq n < m$
- MWe3a: $a^{x_0}ta^{x_1}t\cdots ta^{x_{m+n}}t^{-n}$, some $1 \leq n < m$
- MWe4a: $t^{-l}a^{x_0}ta^{x_1}t\cdots ta^{x_{m+n+l}}t^{-n}$, some $n, l \geq 1$, $n + l < m$.

Words in MWe1a are already of the required form. Cyclic permutation ensures that words in MWe2a or MWe4a are conjugate to words in MWe3a. Such a word can be expressed as follows:

$$a^{x_0}ta^{x_1}t\cdots ta^{x_{m+n}}t^{-n} = a^{x_0}ta^{x_1}t\cdots ta^{\sum_{j=0}^{n} x_{m+j}k^j}$$

by expressing the suffix $a^{x_m}ta^{x_{m+1}}t\cdots a^{x_{m+n}}t^{-n}$ in terms of a only. This is in the required form.

Next, we show that any element of the form $a^{x_0}ta^{x_1}t\cdots a^{x_{m-1}}ta^{x_m}$ is conjugate to an element of the form $a^{y_0}ta^{y_1}t\cdots a^{y_{m-1}}t$, where $|y_i| \leq r$ for all i. To see this, consider the following procedure:

1. Choose $i < m$ such that $|x_i| > r$. Modify the word using the rewrite $a^{\pm(r+1)}t \mapsto a^{\mp(r-1)}ta^{\pm 1}$ (which doesn't change the group element). Repeat this step until there is no such i.

2. Cyclically permute the (now possibly altered) a^{x_m} to the front of the word (which doesn't change the conjugacy class). If there is now some $i < m$ with $|x_i| > r$, return to step 1. Otherwise the procedure terminates.

Clearly if this process terminates we will have a word in the desired form. To see that it does indeed terminate, consider the quantity $h := \sum_{i=0}^{m} |x_i|$. The rewrite in step 1, applied to a^{x_i} say, reduces $|x_i|$ by 2, and modifies $|x_{i+1}|$ by ± 1, depending on signs, thus step 1 always reduces h. Step 2 cannot increase h, it either keeps it constant or reduces it, depending on the signs of x_0 and x_m. Since h can never be negative, the process must terminate.

Finally, we show that any element of the form $a^{y_0}ta^{y_1}t\cdots a^{y_{m-1}}t$, where $|y_i| \leq r$ for all i, is conjugate to an element of $\mathcal{A}_+$. Consider the following procedure:

1. If there are any is with $x_{i-1} = \pm r$ and $\pm x_i < 0$, rewrite the left-most occurrence according to the rule $a^{\pm r}ta^{x_i}t \mapsto a^{\mp r}ta^{x_i \pm 1}t$. Repeat this step until there are no such i.

2. If there are any subwords of the form $a^{\pm r}ta^{\pm r}t$, rewrite the left-most such subword to $a^{\mp r}ta^{\mp(r-1)}ta^{\pm 1}$. Repeat this step until there are no such subwords.

3. If the previous steps have resulted in a new $a^{\pm 1}$ appearing at the end of the word, cyclically permute it to the front. Return to step 1.

It is clear that if this process terminates, the new word will either be an element of $\mathcal{A}_+$, or will be in the set $\{(a^{-r+1}ta^{-r}t)^{\frac{m}{2}}, (a^{-r}ta^{-r+1}t)^{\frac{m}{2}} \mid m \geq 2, m \equiv 0 \pmod 2\}$. In the latter case, the word is conjugate to an element of $\mathcal{A}_+$. This finishes the proof.

To see that the process terminates, note that since we work from left to right, each step will only be repeated a finite number of times before moving onto the next step. Furthermore, working left to right in step 1 also ensures that no additional candidates for step 2 are created. Repeating step 2 any number of times will result in at most one $a^{\pm 1}$ appearing at the right hand end of the word. After cyclically permuting, and returning to step 1, there may be a subword of the form $a^{\pm r}ta^{\pm r}t$ at the start of the word. However, after repeating step 2 as many times as necessary, any letter appearing at the right hand end of the word will have the same sign at the previous time, and thus when cyclically permuted cannot result in another subword of the form $a^{\pm r}ta^{\pm r}t$. Thus the process will terminate. □

5.3 The conjugacy growth series of $BS(1, k)$

In this section we show, in Corollary 5.3.2, that the conjugacy growth series of $BS(1, k)$ with respect to its standard generating set is transcendental. This follows from determining the asymptotics (and transcendental behaviour) of conjugacy growth outside $\mathbb{Z}_k$ in the following proposition.

Proposition 5.3.1. *The generating function for the number of conjugacy classes in $BS(1, k)$ of the form $[(x, m)]$, with $m \neq 0$, is transcendental.*

Proof. We compute the asymptotics for the number of conjugacy classes of length n in $BS(1, k)$ by finding the growth of the set $\mathcal{A}$ in Corollary 5.2.6. We show that this growth is asymptotically of the form $\frac{\rho^n}{n}$ (up to multiplicative constants). In order to obtain this growth expression, we will not count the set $\mathcal{A}$ exactly, but make simplifications along the way which will not affect the final result but will make the counting easier.

We start with the odd case $k = 2r + 1$ and apply Corollary 5.2.6 (1). Since there is a length-preserving bijection between $\mathcal{A}_+$ and $\mathcal{A}_-$, it suffices to consider the asymptotics for $\mathcal{A}_+$. Moreover, since the set $\mathcal{N}_o = \{(a^{-r}t)^m \mid m \geq 1\}$ being removed has negligible size (there is at most one word in $\mathcal{N}_o$ of length n for fixed r and n), it is sufficient to compute the growth of $\mathcal{A}_o := \{a^{x_0}ta^{x_1}t \cdots a^{x_{m-1}}t \mid m \geq 1, |x_i| \leq r\}$. Let $\mathcal{S}_o := \{t, at, a^{-1}t, a^2t, a^{-2}t, \dots, a^rt, a^{-r}t\}$. Then $\mathcal{A}_o$ is equal to $\mathcal{S}_o^*$, so the generating function for $\mathcal{A}_o$ is $\mathcal{A}_o(z) = \frac{1}{1-\mathcal{S}_o(z)}$, where

$$\mathcal{S}_o(z) = z + 2z^2 + \cdots + 2z^{r+1} = z + 2z^2\frac{1-z^r}{1-z} \tag{5.7}$$

is the generating function of $\mathcal{S}_o$ (see Theorem I.1 of [31]). We get

$$\mathcal{A}_o(z) = \frac{1}{1 - z - 2z^2\frac{1-z^r}{1-z}} = \frac{1-z}{1-2z-z^2+2z^{r+2}}. \tag{5.8}$$

The denominator of $\mathcal{A}_o(z)$, that is, the polynomial $p(z) = 1-2z-z^2+2z^{r+2}$, satisfies $p(0) = 1 > 0$ and $p(\frac{1}{2}) < 0$ (and $p(\frac{1}{2}) = 0$ for $r = 1$), so it has a root $\rho_o \in (0, \frac{1}{2})$ (and $\rho_o = \frac{1}{2}$ for $r = 1$). Moreover, $p'(\alpha) = -2 - 2\alpha + 2(r+2)\alpha^{r+1} < 0$ for $0 < \alpha < \frac{1}{2}$, so ρ_o is a simple root. Also, $1 - 2z - z^2 + 2z^{r+2} = (1 - z^2) - 2z(1 - z^{r+1})$, so it has no root in $(-1, 0)$. Thus the growth rate of the set $\mathcal{A}_o$ is $\frac{1}{\rho_o} > 2$, which implies that the number of words of length n in $\mathcal{A}_o$, and therefore also $\mathcal{A}$, is asymptotically $c_o(r)\rho_o^{-n}$, where $c_o(r)$ is a constant depending on r.

Now let k be even, $k = 2r$. The counting is similar, except that we impose on the set $\mathcal{A}_o := \{a^{x_0}ta^{x_1}t \cdots a^{x_{m-1}}t \mid m \geq 1, |x_i| \leq r\}$ considered above the conditions from Corollary 5.2.6(2), that is, a^rt and $a^{-r}t$ can each be followed only by r words out of the total $2r + 1$ in $\mathcal{S}$. Call the set with these restrictions $\mathcal{A}_e$, and let $\mathcal{S}_e = \{t, at, a^{-1}t, \dots, a^{r-1}t, a^{-r+1}t, a^{\pm r}tt, a^{\pm r}ta^{\pm 1}t, \dots, a^{\pm r}ta^{\pm(r-1)}t\}$
(and $\mathcal{S}_e = \{t, a^{\pm 1}tt\}$ for $r = 1$). Note that $\mathcal{S}_e^*$ does not include any words that end in a^rt or $a^{-r}t$, but since we need to consider the set $\mathcal{A}_e$ up to cyclic permutations, the set $\mathcal{S}_e^*$ will in fact suffice to give the asymptotics for $\mathcal{A}_e$ up to cyclic permutations, since it ensures only 'legal' occurrences of a^rt or $a^{-r}t$ appear when cyclically permuting the words.

Then since

$$\mathcal{S}_e(z) = z + 2z^2 + \cdots + 2z^r + 2z^{r+2} + \cdots + 2z^{2r+1} \tag{5.9}$$

$$= z + 2z^2\frac{z^{2r}-1}{z-1} - 2z^{r+1} = \frac{-z - z^2 + 2z^{r+1} - 2z^{r+2} + 2z^{2r+2}}{z-1}$$

we have

$$\mathcal{S}_e^* = \frac{1-z}{1 - 2z - z^2 + 2z^{r+1} - 2z^{r+2} + 2z^{2r+2}}. \tag{5.10}$$

For $r > 1$, the denominator $p(z)$ of (5.10) satisfies $p(0) = 1 > 0$ and $p(\frac{1}{2}) < 0$, so it has a root $\rho_e \in (0, \frac{1}{2})$. Moreover, $p'(\alpha) = -2 - 2\alpha + 2(r+1)\alpha^r - 2(r+2)\alpha^{r+1} + 2(2r+2)\alpha^{2r+1} < 0$ for $0 < \alpha < \frac{1}{2}$, so ρ_e is a simple root, and the growth of the languages $\mathcal{S}_e^*$, and consequently $\mathcal{A}_e$, is $\frac{1}{\rho_e} > 0$. (For $r = 1$, $\rho_e \approx 0.590$.) Also, $1 - 2z - z^2 + 2z^{r+1} - 2z^{r+2} + 2z^{2r+2} = (1 - z^2) - 2z(1 + z^{r+1})(1 - z^r)$, so it has no root in $(-1, 0)$. This implies that the number of words of length n in $\mathcal{A}_e$, is asymptotically $c_e(r)\rho_e^{-n}$, where $c_e(r)$ is a constant depending on r.

Now in order to find the growth of the conjugacy classes for $m \neq 0$, we need to count the number of representatives of length n in $\mathcal{A}_o$ or $\mathcal{A}_e$, up to the cyclic permutation of the subwords in $\mathcal{S}_o$ or $\mathcal{S}_e$. For each word in $\mathcal{A}_o$ or $\mathcal{A}_e$ there are m possible distinct cyclic permutations unless that word is a non-trivial power. Given that the number of powers is negligible compared to the total number of words (to see this, suppose that an alphabet X consists of $x > 1$ letters; while the number of words of length n over X is x^n, the number of proper powers of length n is $\sum_{d|n} x^d < \sum_{i=1}^{n/2} x^i < x^{n/2+1}$, and $\frac{x^{n/2+1}}{x^n} \to 0$ as $n \to \infty$), for fixed n and m the number of cyclic representatives of words in $\mathcal{A}_o$ and $\mathcal{A}_e$ is approximately $c_o(r)\frac{\rho_o^{-n}}{m}$ and $c_e(r)\frac{\rho_e^{-n}}{m}$, respectively. Since each word of length n in $\mathcal{A}_o$ or $\mathcal{A}_e$ consists of m 'syllables' of bounded length we get $\frac{n}{r+1} \leq m \leq n$ in the odd case and $\frac{n}{r+\frac{1}{2}} \leq m \leq n$ in the even case, so the number a_n^o of cyclic representatives in the odd case satisfies

$$c_o(r)\frac{\rho_o^{-n}}{n} \leq a_n^o \leq c_o(r)\frac{(r+1)\rho_o^{-n}}{n} \tag{5.11}$$

and in the even case the number a_n^e of cyclic representatives satisfies

$$c_e(r)\frac{\rho_e^{-n}}{n} \leq a_n^e \leq c_e(r)\frac{(r+\frac{1}{2})\rho_e^{-n}}{n} \tag{5.12}$$

Finally, by [30, Theorem D] the generating function for any sequence with asymptotics of the form (5.11) or (5.12), that is, bounded on both sides by terms $\frac{\rho^n}{n}$(up to multiplicative constants), is transcendental. $\square$

Corollary 5.3.2. *The conjugacy growth series for $BS(1,k)$, with respect to the generating set $\{a,t\}$, is transcendental.*

Proof. By Propositions 5.1.2, 5.1.6, 5.1.9, the conjugacy growth series for $\mathbb{Z}_k$ (when $m=0$) is rational, and by Proposition 5.3.1 the generating function for conjugacy classes of the form $[(x,m)]$ with $m \neq 0$ is transcendental. Since the sum of a transcendental function and a rational function is transcendental, we obtain the result. $\square$

Corollary 5.3.3. *The conjugacy and standard growth rates of $BS(1,k)$, with respect to the generating set $\{a,t\}$, are equal.*

Proof. We start with the odd case. By [17, Theorem (iii)] (see also [7, Lemma 11(b)]) the standard growth rate is the inverse of the smallest absolute value of the real roots of the polynomial $1-2t-t^2+2t^{r+2}$ which appears in the denominator of the standard growth series. But the same polynomial appears in the denominator of (5.8), and since the smallest absolute value of real roots is $\rho_o \leq \frac{1}{2}$, this will dominate the growth rate of the conjugacy classes in $\mathbb{Z}_k$, which is smaller than 2 by Corollary 5.1.4. Thus the standard and the conjugacy growth rates are equal.

In the even case with $k>2$, note that the second factor in the denominator in [17, Theorem (i)] is identical to that in formula (5.10), and both denominators have the same smallest absolute value of real roots $\rho_e < \frac{1}{2}$ which dominates the growth rate of the conjugacy classes in $\mathbb{Z}_k$, which is smaller than 2 by Corollary 5.1.7, so the two rates are equal.

In the case when $k=2$, note that the second factor in the denominator in [17, Theorem (ii)] is also a factor to that in formula (5.10), and both denominators have the same smallest absolute value of real roots $\rho_e \approx 0.590$ which dominates the growth rate of the conjugacy classes in $\mathbb{Z}_k$, which is approximately $\frac{1}{0.742} \approx 1.348$ by Corollaries 5.1.10, so the two rates are equal. $\square$

We note that it follows from work of Valiunas [58] that the relative standard growth of the subgroup $\mathbb{Z}_k$ is bounded above by the standard growth of $BS(1,k)\backslash\mathbb{Z}_k$. Since conjugacy growth is bounded above by standard growth, this is sufficient to prove Corollary 5.3.3, without using the specific bounds for $\mathbb{Z}_k$ we computed in Corollaries 5.1.4 and 5.1.7. However, the computation of those bounds adds to the quantitative understanding of the conjugacy growth asymptotics and the formulae for the conjugacy growth series of $\mathbb{Z}_k$ are necessary for the computations in the next section.

5.4 Conjugacy growth series formulae

In this section we give formulae for the growth series of the conjugacy classes of $BS(1,k)$ outside the normal abelian subgroup $\mathbb{Z}_k$. That is, we compute the generating function for the set $\mathcal{A}$, up to cyclic permutation, given in Corollary 5.2.6.

In the description of $\mathcal{A}$ in Corollary 5.2.6 there is a length-preserving bijection between $\mathcal{A}_+$ and $\mathcal{A}_-$, so it suffices to consider the generating function for the set $\mathcal{A}_+$ up to cyclic permutations.

In the odd $k = 2r+1$ case, as the set $\mathcal{N}_o = \{(a^{-r}t)^m \mid m \geq 1\}$ has generating function $\mathcal{N}_o(z) = \sum_{m\geq 1} z^{(r+1)m}$, it is sufficient to compute the generating function of $\mathcal{A}_o := \{a^{x_0}ta^{x_1}t\cdots a^{x_{m-1}}t \mid m \geq 1, |x_i| \leq r\}$ up to cyclic permutation.

In the $k = 2r$ case as the set $\mathcal{N}_e = \{(a^{-r+1}ta^{-r}t)^{\frac{m}{2}}, (a^{-r}ta^{-r+1}t)^{\frac{m}{2}} \mid m \geq 2, m \equiv 0 \pmod 2\}$ has generating function $\mathcal{N}_e(z) = \sum_{m\geq 1} z^{(2r+1)m}$, it is sufficient to compute the generating function of $\mathcal{A}_e = \{a^{x_0}ta^{x_1}t\cdots a^{x_{m-1}}t \mid m \geq 1, |x_i| \leq r, \forall i(x_{i-1} = \pm r \implies 0 \leq \pm x_i < r)\}$ up to cyclic permutation.

This is exactly the *cycle construction* (see page 26 in [31]) applied to the sets

$$\mathcal{S}_o = \{t, at, a^{-1}t, a^2t, a^{-2}t, \ldots, a^rt, a^{-r}t\}$$

and

$$\mathcal{S}_e = \{t, at, a^{-1}t, \ldots, a^{r-1}t, a^{-r+1}t, a^{\pm r}tt, a^{\pm r}ta^{\pm 1}t, \ldots, a^{\pm r}ta^{\pm(r-1)}t\},$$

respectively, defined in the proof of Proposition 5.3.1. Thus by applying the formula in [31, Theorem I.1], we get that

$$Cyc(\mathcal{A}_o)(z) = \sum_{k=1}^{\infty} \frac{-\phi(k)}{k} \log(1 - \mathcal{S}_o(z^k)), \tag{5.13}$$

where $\mathcal{S}_o(z)$ is given in (5.7), and ϕ is the Euler totient function. In the odd case we get

$$Cyc(\mathcal{A}_e)(z) = \sum_{k=1}^{\infty} \frac{-\phi(k)}{k} \log(1 - \mathcal{S}_e(z^k)), \tag{5.14}$$

where $\mathcal{S}_e(z)$ is given in (5.9). We thus obtain (note $\mathcal{S}_o$ and S_o etc are different):

Proposition 5.4.1. *The conjugacy growth series for $BS(1, 2r + 1)$ is the series*

$$S_o(z) + Cyc(\mathcal{A}_o) - \mathcal{N}_o(z),$$

where $S_o(z)$ is given by (5.3), and the conjugacy growth series for $BS(1, 2r)$ is

$$S_e(z) + Cyc(\mathcal{A}_e) - \mathcal{N}_e(z),$$

where $S_e(z)$ is given by (5.4).

Chapter 6

Conjectures and future work

6.1 Rational Conjugacy Growth

We saw in Theorem 3.3.1 that virtually abelian groups always have rational conjugacy growth series. Every conjugacy growth series for a non-virtually abelian finitely presented group which has been studied, has been found to be transcendental. This leads us to make the following conjecture.

Conjecture 6.1.1 (see also [26]). *The conjugacy growth series of any finitely presented group that is not virtually abelian is transcendental.*

Note that we include finitely *presented* in the hypothesis as Hull and Osin have constructed a group with exponential conjugacy growth which possesses an index 2 subgroup with only two conjugacy classes. So this subgroup has rational growth (in fact, the growth series is simply a polynomial). However, this subgroup is not finitely presented.

6.2 Growth Rates

Regarding growth rates, we ask the following question.

Question 6.2.1. *If the conjugacy and standard growth rates of a group are equal for some generating set, are they equal for all generating sets?*

The question is related to the conjecture below. This has already been shown to hold in hyperbolic [1], relatively hyperbolic [33], most graph products [14], and lamplighter groups [49].

Conjecture 6.2.2. *For any choice of generating set, the conjugacy and standard growth rates of a finitely presented group are equal.*

6.3 Quasi-Isometric invariance

Definition 6.3.1. Let (X, d) and (X', d') be metric spaces. Then a map $f\colon (X, d) \to (X', d')$ is a *quasi-isometric embedding* if there exist constants $\lambda \geq 1$ and $C \geq 0$ such that

$$\frac{1}{\lambda}d(x, y) - C \leq d'(f(x), f(y)) \leq \lambda d(x, y) + C$$

for all $x, y \in X$. If, in addition, there is a constant $D \geq 0$ such that for all $z \in X'$, there is some $w \in f(X)$ with $d'(x, w) \leq D$, then f is a *quasi-isometry.* We say that X and X' are *quasi-isometric.*

Any two Cayley graphs of the same finitely generated group are quasi-isometric. This allows us to extend the definition above to groups. We say that groups G and H are quasi-isometric if any (and therefore every) pair of their respective Cayley graphs are quasi-isometric. There is considerable interest in identifying which properties of groups are invariant under quasi-isometry. It is not hard to see that the asymptotic behaviour of the standard growth function is such a property.

As mentioned above, Hull and Osin [45] exhibit a finitely generated group with exponential conjugacy growth function, possessing a finite-index subgroup with only two conjugacy classes. Since any group is quasi-isometric to its finite-index subgroups, this demonstrates that the conjugacy growth type of a group is not a quasi-isometry invariant. On the other hand, the conjugacy growth of a virtually abelian group is equivalent to its standard growth (see Proposition 3.0.1), and so conjugacy growth is a quasi-isometry invariant in this very restricted context. The question arises then as to which other groups fall into this category. A natural next class of groups to consider would be the (virtually) nilpotent groups. The following Theorem of Pansu gives insight into the nature of quasi-isometry in nilpotent groups.

Theorem 6.3.2 ([27],[51])**.** *For a finitely generated nilpotent group G, the numbers* $\mathrm{rk}\left(G^{(i+1)}/G^{(i)}\right)$ *are quasi-isometry invariants.*

With this in mind, we venture the following conjecture, which would imply that conjugacy growth was a quasi-isometry invariant in this class of groups.

Conjecture 6.3.3. *The conjugacy growth of a finitely generated nilpotent group G depends only on the numbers* $\mathrm{rk}\left(G^{(i+1)}/G^{(i)}\right)$*, i.e. the torsion-free ranks of the quotients of the lower central series.*

6.4 Further Work

We conclude with some concrete research directions that arise directly from the work in this thesis.

1. Theorem 4.3.3 shows that certain groups with a finite-index subgroup isomorphic to the higher Heisenberg groups H_r have conjugacy growth equivalent to that of H_r. Theorem 4.5.1 shows that all virtually H_1 groups have growth equivalent to that of H_1. To show that *all* virtually H_r groups have growth equivalent to H_r, it should be possible to generalise Theorem 4.5.1, considering finite order automorphisms of H_r.

2. With a view to proving Conjecture 6.3.3, it would be useful to have a larger amount of data about the asymptotics of conjugacy growth in nilpotent groups. A good place to start would be to improve the bound for free nilpotent groups of class 2 given in Proposition 4.6.2.

3. Sánchez and Shapiro [55] study the growth of so-called *higher Baumslag-Solitar* groups. These are HNN-extensions of $\mathbb{Z}^m$ given by the endomorphism $g \mapsto g^3$ that cubes all elements. Formal language techniques are used to show that these groups have rational growth series, with respect to a natural choice of generating set. Some of the techniques of Chapter 5 may allow us to study conjugacy growth in this setting and help verify conjecture 6.1.1 for this special case.

4. For groups with transcendental conjugacy growth, it would be interesting to classify their algebraic complexity more finely. For example, a series is called *D-finite* if it is the solution to an equation involving derivatives of finite order,

as well as polynomials. All algebraic series are D-finite, but not vice-versa. It would be interesting to investigate which groups admit D-finite conjugacy growth series.

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